MATH 559 - ASSIGNMENT 1–SOLUTIONS

For $n \geq 1$, suppose the de Finetti representation for the joint pmf of exchangeable discrete random variables $Y_1, \ldots, Y_n$ is given by

$$p_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = \int_\Theta \prod_{i=1}^n p_Y(y_i;\theta)\pi_0(d\theta)$$

where $p_Y(y;\theta)$ is a mass function in y, and θ is a parameter lying in a space $\Theta \subseteq \mathbb{R}^p$, for some (prior) distribution $\pi_0(d\theta)$ defined on Θ, where we may interpret Θ as the smallest set such that

$$\int_\Theta \pi_0(d\theta) = 1.$$

As pointed out by Bernardo & Smith, the Poisson model with mass function

$$p_Y(y;\theta) = \frac{\theta^y \exp\{-\theta\}}{y!} \qquad y = 0, 1, 2, \ldots$$

and zero otherwise, for parameter $\theta > 0$, arises by considering observables taking values on the non-negative integers that yield certain summary statistics, or as the limiting case of a discrete selection (multinomial) model.

For this Poisson model:

(a) *Find the form of $p_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n)$ if π_0 is the Gamma density with parameters (α_0, β_0), that is*

$$\pi_0(d\theta) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\theta^{\alpha_0-1}\exp\{-\beta_0\theta\}d\theta$$

and $\Theta = \mathbb{R}^+ \equiv \{t \in \mathbb{R} : t > 0\}$.

Solution: We have from above that, for any $n \geq 1$, and any vector of non-negative integers $(y_1, \ldots, y_n)$, the prior predictive (joint) mass function is

$$\begin{aligned}
p_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) &= \int_0^\infty \prod_{i=1}^n p_Y(y_i;\theta)\pi_0(\theta)\, d\theta \\
&= \int_0^\infty \prod_{i=1}^n \frac{\theta^{y_i}\exp\{-\theta\}}{y_i!} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\theta^{\alpha_0-1}\exp\{-\beta_0\theta\}\, d\theta \\
&= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{1}{\prod\limits_{i=1}^n y_i!} \int_0^\infty \theta^{s_n+\alpha_0-1}\exp\{-(n+\beta_0)\theta\}\, d\theta \qquad s_n = \sum_{i=1}^n y_i \\
&= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{1}{\prod\limits_{i=1}^n y_i!} \frac{\Gamma(s_n+\alpha_0)}{(n+\beta_0)^{(s_n+\alpha_0)}}.
\end{aligned}$$

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Note: the implied distribution of $S_n = \sum_{i=1}^n Y_i$ is known as the *Poisson-Gamma* distribution.

(b) *For the choice of π_0 in (a), compute the implied (marginal) covariance between Y_1 and Y_2.*

Solution: We have that for $i = 1, \ldots, n$, by iterated expectation

$$\mathbb{E}_{Y_i}[Y_i] = \mathbb{E}_{\pi_0}[\mathbb{E}_{Y_i|\theta}[Y_i|\theta]]$$

and

$$\mathbb{E}_{Y_i}[Y_i^2] = \mathbb{E}_{\pi_0}[\mathbb{E}_{Y_i|\theta}[Y_i^2|\theta]] = \mathbb{E}_{\pi_0}[\mathrm{Var}_{Y_i|\theta}[Y_i|\theta] + \{\mathbb{E}_{Y_i|\theta}[Y_i|\theta]\}^2]$$

so combining terms together

$$\mathrm{Var}_{Y_i}[Y_i] = \mathbb{E}_{Y_i}[Y_i^2] - \{\mathbb{E}_{Y_i}[Y_i]\}^2 = \mathbb{E}_{\pi_0}[\mathrm{Var}_{Y_i|\theta}[Y_i|\theta]] + \mathbb{E}_{\pi_0}[\{\mathbb{E}_{Y_i|\theta}[Y_i|\theta]\}^2 - \{\mathbb{E}_{Y_i}[Y_i]\}^2]$$

as the term $\mathbb{E}_{Y_i}[Y_i]$ does not depend on θ. Thus, by the original iterated expectation result, the variance can be written

$$\mathrm{Var}_{Y_i}[Y_i] = \mathbb{E}_{\pi_0}[\mathrm{Var}_{Y_i|\theta}[Y_i|\theta]] + \mathrm{Var}_{\pi_0}[\mathbb{E}_{Y_i|\theta}[Y_i|\theta]]$$

which is known as the *iterated variance formula*. Thus, by properties of the Poisson distribution

$$\mathrm{Var}_{Y_i}[Y_i] = \mathbb{E}_{\pi_0}[\theta] + \mathrm{Var}_{\pi_0}[\theta]$$

and by properties of the Gamma distribution, we have finally

$$\mathrm{Var}_{Y_i}[Y_i] = \frac{\alpha_0}{\beta_0} + \frac{\alpha_0}{\beta_0^2} = \frac{\alpha_0(1+\beta_0)}{\beta_0^2}.$$

For the covariance, using iterated expectation we have

$$\begin{aligned}
\mathbb{E}_{Y_i,Y_j}[Y_iY_j] &= \mathbb{E}_{\pi_0}\left[\mathbb{E}_{Y_i,Y_j|\theta}[Y_iY_j|\theta]\right] \\
&= \mathbb{E}_{\pi_0}\left[\mathbb{E}_{Y_i|\theta}[Y_i|\theta]\mathbb{E}_{Y_j|\theta}[Y_j|\theta]\right] && Y_i, Y_j \text{ cond. indep.} \\
&= \mathbb{E}_{\pi_0}[\theta^2] \\
&= \mathrm{Var}_{\pi_0}[\theta] + \{\mathbb{E}_{\pi_0}[\theta]\}^2.
\end{aligned}$$

Thus

$$\mathrm{Cov}_{Y_i,Y_j}[Y_i, Y_j] = \mathbb{E}_{Y_i,Y_j}[Y_iY_j] - \mathbb{E}_{Y_i}[Y_i]\mathbb{E}_{Y_j}[Y_j] = \mathrm{Var}_{\pi_0}[\theta] = \frac{\alpha_0}{\beta_0^2}.$$

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(c) *Suppose that a discrete prior is chosen, where the corresponding mass function takes the form*

$$\pi_0(\theta) = \frac{1}{3}\mathbb{1}_{\{1\}}(\theta) + \frac{2}{3}\mathbb{1}_{\{2\}}(\theta)$$

that is, the prior places probability 1/3 on the value 1, and 2/3 on the value 2. Compute the implied (marginal) covariance between Y_1 and Y_2 for this prior.
Hint: in this case, $\Theta \equiv \{1, 2\}$, and the integral in the de Finetti representation reduces to a sum.

Solution: Here, we can use the same methodology as above immediately to deduce the forms

$$\begin{aligned}
\mathbb{E}_{Y_i}[Y_i] &= \mathbb{E}_{\pi_0}[\theta] \\
\mathrm{Var}_{Y_i}[Y_i] &= \mathbb{E}_{\pi_0}[\theta] + \mathrm{Var}_{\pi_0}[\theta] \\
\mathrm{Cov}_{Y_i,Y_j}[Y_i, Y_j] &= \mathrm{Var}_{\pi_0}[\theta].
\end{aligned}$$

For this discrete prior, we have

$$\mathbb{E}_{\pi_0}[\theta] = \frac{1}{3} \times 1 + \frac{2}{3} \times 2 = \frac{5}{3}$$

and

$$\mathbb{E}_{\pi_0}[\theta^2] = \frac{1}{3} \times 1 + \frac{2}{3} \times 4 = 3$$

so

$$\text{Var}_{\pi_0}[\theta] = 3 - \left(\frac{5}{3}\right)^2 = \frac{2}{9}.$$

Hence

$$\begin{aligned}
\mathbb{E}_{Y_i}[Y_i] &= \frac{5}{3} \\
\text{Var}_{Y_i}[Y_i] &= \frac{5}{3} + \frac{2}{9} = \frac{17}{9} \\
\text{Cov}_{Y_i,Y_j}[Y_i, Y_j] &= \frac{2}{9}
\end{aligned}$$

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(d) *For the prior in (a), find the posterior predictive distribution for Y_3 given $Y_1 = y_1$ and $Y_2 = y_2$.*

Solution: By definition, we have that the posterior predictive distribution is given by

$$p_{Y_3|Y_1,Y_2}(y_3|y_1, y_2) = \frac{p_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)}{p_{Y_1,Y_2}(y_1, y_2)}$$

provided the denominator is non-zero at the required arguments. Using the above form from (a), we deduce that

$$\begin{aligned}
p_{Y_3|Y_1,Y_2}(y_3|y_1, y_2) &= \frac{\dfrac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \dfrac{1}{\prod\limits_{i=1}^{3} y_i!} \dfrac{\Gamma(s_3 + \alpha_0)}{(3 + \beta_0)^{(s_3+\alpha_0)}}}{\dfrac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \dfrac{1}{\prod\limits_{i=1}^{2} y_i!} \dfrac{\Gamma(s_2 + \alpha_0)}{(2 + \beta_0)^{(s_2+\alpha_0)}}} \\
&= \frac{1}{y_3!} \frac{\Gamma(s_3 + \alpha_0)}{\Gamma(s_2 + \alpha_0)} \frac{(2 + \beta_0)^{(s_2+\alpha_0)}}{(3 + \beta_0)^{(s_3+\alpha_0)}}
\end{aligned}$$

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