In this MidTerm there are three questions:

- Q1: 20 Marks
- Q2: 20 Marks
- Q3: 10 Marks

Q1. Consider realization $Y_1, \ldots, Y_n$ from an infinitely exchangeable binary sequence where, in de Finetti's representation, prior distribution $\pi_0(\theta) \equiv Beta(\alpha_0, \beta_0)$ on the parameter $\theta$ is specified. Let $y_1, \ldots, y_n$ be the corresponding observed values, and let

$$s_n = \sum_{i=1}^{n} y_i.$$

The posterior distribution will be denoted $\pi_n$.

(a) Find $\pi_n(\theta)$ if $\alpha_0 = 1/2, \beta_0 = 1$ and $n = 10$, $s_n = 8$. 6 Marks

(b) Find the Bayesian estimate of $\theta$ if the loss function

$$L(t, \theta) = \frac{(t-\theta)^2}{\sqrt{\theta(1-\theta)}}$$

is used for the setting in (a). 4 Marks

(c) Find the posterior predictive distribution, $p_n(y)$, for a single future data point. 6 Marks

(d) If $n$ is large, write down a suitable Normal approximation to the posterior distribution. 4 Marks

**Solution:**

(a) We have that the likelihood is

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{s_n}(1-\theta)^{n-s_n}$$

and with a $Beta(1/2, 1)$ prior, this means that 6 Marks

$$\pi_n(\theta) \propto \theta^{8+1/2-1}(1-\theta)^{2+1-1} \equiv Beta(8.5, 3) \equiv Beta(\alpha_n, \beta_n)$$

(b) For the prior in (a), we wish to minimize

$$\int_0^1 (t-\theta)^2 \frac{1}{\theta^{1/2}(1-\theta)^{1/2}} \frac{\Gamma(\alpha_n+\beta_n)}{\Gamma(\alpha_n)\Gamma(\beta_n)} \theta^{\alpha_n-1}(1-\theta)^{\beta_n-1}\, d\theta$$

or equivalently

$$\frac{\Gamma(\alpha_n+\beta_n)}{\Gamma(\alpha_n)\Gamma(\beta_n)} \int_0^1 (t-\theta)^2 \theta^{\alpha_n-1/2-1}(1-\theta)^{\beta_n-1/2-1}\, d\theta$$

and by the usual argument, we see that the minimizer is the expectation of the $Beta(\alpha_n - 1/2, \beta_n - 1/2)$ density, that is 4 Marks

$$\frac{\alpha_n - 1/2}{\alpha_n + \beta_n - 1} = \frac{8}{10.5}.$$

(c) We have that for $y = 0, 1$

$$p_n(y) = \int_0^1 f_Y(y;\theta)\pi_n(\theta)\, d\theta = \int_0^1 \theta^y(1-\theta)^{1-y} \frac{\Gamma(11.5)}{\Gamma(8.5)\Gamma(3)} \theta^{8.5-1}(1-\theta)^{3-1}\, d\theta$$

$$= \frac{\Gamma(11.5)}{\Gamma(8.5)\Gamma(3)} \frac{\Gamma(8.5+y)\Gamma(3+1-y)}{\Gamma(12.5)} = \begin{cases} \dfrac{3}{11.5} & y = 0 \\ \dfrac{8.5}{11.5} & y = 1 \end{cases}$$

and zero otherwise. 6 Marks

(d) The first and second derivatives of the log mass function are

$$\dot{\ell}(\theta) = \frac{y}{\theta} - \frac{(1-y)}{1-\theta)} \qquad \ddot{\ell}(\theta) = -\frac{y}{\theta^2} - \frac{(1-y)}{(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

from which we deduce that $\widehat{\theta}_n = s_n/n$, and can assess the the approximation as $\pi_n(\theta) \doteq Normal(\widehat{\theta}_n, \widehat{\theta}_n(1-\widehat{\theta}_n)/n)$ 4 Marks

Q2. A Bayesian analysis based on the conditional model

$$f_Y(y;\theta) \equiv Normal(\theta, \sigma^2)$$

with $\sigma^2$ known is to be carried out based on a random sample of size $n$.

(a) Find the Jeffreys prior for this problem. 6 Marks

(b) A prior for this problem is specified as

$$\pi_0(\theta) \equiv Normal(2, \sigma^2)$$

Find the posterior mean under this specification. 4 Marks

(c) Find the form of the prior predictive distribution (or marginal likelihood) for $Y_1, \ldots, Y_n$ based on the prior in (b). 6 Marks

(d) Explain how to construct a 95% Bayesian credible interval for $\theta$. 4 Marks

**Solution:**

(a) Here we have, in the usual notation,

$$\ell(y;\theta) = \text{constant} - \frac{1}{2\sigma^2}(y-\theta)^2$$

$$\dot{\ell}(y;\theta) = \frac{1}{\sigma^2}(y-\theta)$$

$$\ddot{\ell}(y;\theta) = -1/\sigma^2 = \text{ constant}$$

so the Jeffreys prior is $\pi_0(\theta) \propto 1$ as in the location family case. 6 Marks

(b) We are faced with the complete-the-square problem

$$\sum_{i=1}^{n}(y_i-\theta)^2 + (\theta-2)^2 = \sum_{i=1}^{n}(y_i-\overline{y})^2 + n(\overline{y}-\theta)^2 + (\theta-2)^2$$

so the critical part is $(n+1)(\theta-m_n)^2$, with $m_n = (n\overline{y}+2)/(n+2)$, with the $\sigma^2$ terms cancelling. Incidentally, the posterior variance is $M_n = \sigma^2/(n+1)$. 4 Marks

(c) We have that

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma^2}\right)^{\frac{(n+1)}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i-\overline{y})^2 + n(\overline{y}-\theta)^2 + (\theta-2)^2\right]\right\} d\theta \\ &= k_n \int_{-\infty}^{\infty} \exp\left\{-\frac{(n+1)}{2\sigma^2}(\theta-m_n)^2\right\} d\theta \\ &= k_n\sqrt{2\pi M_n} \end{aligned}$$

where

$$k_n = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{(n+1)}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(y_i-\overline{y})^2 + \frac{n}{n+1}(\overline{y}-2)^2\right]\right\}$$

6 Marks

(d) By inspecting the posterior distribution, which in this case is Normal, we find an interval that contains 95% probability: for example, an equal tail interval uses the 0.025 and 0.975 quantiles of the $Normal(m_n, M_n)$ distribution. 4 Marks

Q3. A Bayesian analysis based on the conditional model

$$f_Y(y;\theta) = c(\theta) y e^{-\theta y} \qquad y > 0$$

and zero otherwise, for $\theta > 0$ and constant $c(\theta)$, is to be carried out based on a random sample of size $n$.

(a) Write down the form of the function $c(\theta)$. 4 Marks

(b) Find the form of the posterior distribution under a conjugate prior. 6 Marks

**Solution:**

(a) By inspection, we see that this is the $Gamma(2, \theta)$ distribution, so

$$c(\theta) = \frac{\theta^2}{\Gamma(2)} = \theta^2$$

4 Marks

(b) If $s_n = y_1 + \cdots + y_n$, the likelihood is then

$$\mathcal{L}_n(\theta) = \left( \prod_{i=1}^{n} y_i \right) \theta^{2n} \exp\{-s_n \theta\}$$

and a conjugate prior is

$$\pi_0(\theta) \equiv \Gamma(\alpha_0, \beta_0)$$

so that the posterior is $\pi_n(\theta) \equiv Gamma(\alpha_n, \beta_n)$ where

$$\alpha_n = 2n + \alpha_0 \qquad \beta_n = s_n + \beta_0.$$

6 Marks