In this MidTerm there are three questions:

- Q1: 20 Marks
- Q2: 20 Marks
- Q3: 10 Marks

Q1. Consider realization $Y_1, \ldots, Y_n$ from an infinitely exchangeable binary sequence where, in de Finetti's representation, prior distribution $\pi_0(\theta) \equiv Beta(\alpha_0, \beta_0)$ on the parameter θ is specified. Let $y_1, \ldots, y_n$ be the corresponding observed values. The posterior distribution will be denoted π_n.

(a) Find $\pi_n(\theta)$ if $\alpha_0 = \beta_0 = 1/2$. 6 Marks

(b) Find the posterior mode for $\pi_n(\theta)$ from (a). 4 Marks

(c) Find the posterior predictive distribution, $p_n(y)$, for a single future data point. 6 Marks

(d) Suppose that ϕ is the quantity

$$\phi = \log\left(\frac{\theta}{1-\theta}\right).$$

Explain how to compute the posterior distribution for ϕ. You do not have to perform the computation. 4 Marks

Solution:

(a) We have that the likelihood is

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{s_n}(1-\theta)^{n-s_n}$$

and with a $Beta(1/2, 1/2)$ prior, this means that 6 Marks

$$\pi_n(\theta) \propto \theta^{s_n+1/2-1}(1-\theta)^{n-s_n+1/2+1} \equiv Beta(s_n + 1/2, n - s_n + 1/2)$$

(b) On the log scale, π_n becomes

$$(s_n - 1/2)\log\theta + (n - s_n - 1/2)\log(1-\theta) = a\log\theta + b\log(1-\theta)$$

say. The derivative of the $\log \pi_n(\theta)$ wrt θ is $a/\theta - b/(1-\theta)$, and equating this to zero to get the mode yields

$$a - a\theta = b\theta \quad \therefore \quad \theta = \frac{a}{a+b} = \frac{s_n - 1/2}{n-1}$$

which is the posterior mode. 4 Marks

(c) We have that for $y = 0, 1$

$$\begin{aligned} p_n(y) &= \int_0^1 f_Y(y;\theta)\pi_n(\theta)\, d\theta \\ &= \int_0^1 \theta^y(1-\theta)^{1-y} \frac{\Gamma(n+1)}{\Gamma(s_n+1/2)\Gamma(n-s_n+1/2)} \theta^{(s_n+1/2)-1}(1-\theta)^{(n-s_n+1/2)-1}\, d\theta \\ &= \frac{\Gamma(n+1)}{\Gamma(s_n+1/2)\Gamma(n-s_n+1/2)} \frac{\Gamma(s_n+y+1/2)\Gamma(n-s_n+1-y+1/2)}{\Gamma(n+2)} \\ &= \begin{cases} \dfrac{n-s_n+1/2}{n+1} & y=0 \\ \dfrac{s_n+1/2}{n+1} & y=1 \end{cases} \end{aligned}$$

and zero otherwise. 6 Marks

(d) This is a simple transformation exercise; $\pi_n(\theta)$ yields the posterior for ϕ including the Jacobian. In this case,

$$\phi = \log\left(\frac{\theta}{1-\theta}\right) \qquad \Longrightarrow \qquad \theta = \frac{e^\phi}{(1+e^\phi)}$$

so that

$$\pi_n(e^\phi/(1+e^\phi))\frac{e^\phi}{(1+e^\phi)^2} \qquad \phi \in \mathbb{R}$$

4 Marks

Q2. A Bayesian analysis based on the Poisson conditional model

$$f_Y(y;\theta) = \frac{e^{-\theta}\theta^y}{y!} \qquad y = 0, 1, 2 \ldots$$

and zero otherwise, for $\theta > 0$, is to be carried out based on a random sample of size n.

(a) A prior distribution $\pi_0(\theta) \equiv Gamma(2, 1)$ is specified. Find the prior predictive distribution (or marginal likelihood) for $Y_1, \ldots, Y_n$. 6 Marks

(b) For the prior in (a), find the Bayes estimate of θ derived under the loss function

$$L(t, \theta) = (t - \theta)^2 e^{-\theta}$$

4 Marks

(c) A prior distribution $\pi_0(\theta)$ that places equal probability on the values 1 and 2 is specified. Find the posterior distribution for θ. 6 Marks

(d) A prior distribution $\pi_0(\theta) \equiv Uniform(0, L)$ is specified. In the limiting case $L \longrightarrow \infty$, this prior is improper (i.e. does not integrate to 1). Is the posterior distribution improper ? Justify your answer. 4 Marks

Solution:

(a) We have that if $s_n = y_1 + \cdots + y_n$, $\alpha_0 = 2$, $\beta_0 = 1$, then

$$\begin{aligned} f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) &= \int_0^\infty \prod_{i=1}^n \frac{e^{-\theta}\theta^{y_i}}{y_i!} \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \theta^{\alpha_0-1} e^{-\beta_0\theta}\, d\theta \\ &= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{1}{\prod_{i=1}^n y_i!} \int_0^\infty \theta^{s_n+\alpha_0-1} e^{-(n+\beta_0)\theta}\, d\theta \\ &= \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)} \frac{1}{\prod_{i=1}^n y_i!} \frac{\Gamma(s_n+\alpha_0)}{(n+\beta_0)^{s_n+\alpha_0}} \end{aligned}$$

which is the prior predictive. 6 Marks

(b) For the prior in (a), the posterior is

$$\pi_n(\theta) \equiv Gamma(s_n+\alpha_0, n+\beta_0) \equiv Gamma(\alpha_n, \beta_n)$$

say, so we wish to minimize

$$\int_0^\infty (t-\theta)^2 e^{-\theta} \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} \theta^{\alpha_n-1} e^{-\beta_n\theta}\, d\theta = \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} \int_0^\infty (t-\theta)^2 \theta^{\alpha_n-1} e^{-(\beta_n+1)\theta}\, d\theta$$

and by the usual argument, we see that the minimizer is the expectation of the $Gamma(\alpha_n, \beta_n+1)$ density, that is 4 Marks

$$\frac{\alpha_n}{\beta_n+1} = \frac{s_n+2}{n+2}.$$

(c) By inspection, the posterior is

$$\pi_n(\theta) = \frac{e^{-n}\mathbb{1}_{\{1\}}(\theta) + e^{-2n}2^{s_n}\mathbb{1}_{\{2\}}(\theta)}{e^{-n}+e^{-2n}2^{s_n}} = \begin{cases} \dfrac{1}{1+e^{-n}2^{s_n}} & \theta = 1 \\ \dfrac{e^{-n}2^{s_n}}{1+e^{-n}2^{s_n}} & \theta = 2 \end{cases}$$

and zero otherwise, that is, concentrating on $\{1,2\}$ 6 Marks

(d) In this case the posterior is proportional to the likelihood

$$\pi_n(\theta) \propto \theta^{s_n} e^{-n\theta}$$

which is integrable as this is then a $Gamma(s_n+1, n)$ distribution, and the posterior is proper. 4 Marks

Q3. A Bayesian analysis based on the conditional model

$$f_Y(y;\theta) = c(\theta)y^2e^{-\theta y} \qquad y > 0$$

and zero otherwise, for $\theta > 0$ and constant $c(\theta)$, is to be carried out based on a random sample of size n.

(a) Write down the form of the function $c(\theta)$. 4 Marks

(b) Find the form of the posterior distribution under a conjugate prior. 6 Marks

Solution:

(a) By inspection, we see that this is the $Gamma(3, \theta)$ distribution, so

$$c(\theta) = \frac{\theta^3}{\Gamma(3)} = \frac{1}{2}\theta^3$$

4 Marks

(b) If $s_n = y_1 + \cdots + y_n$, the likelihood is then

$$\mathcal{L}_n(\theta) = \left(\prod_{i=1}^{n} y_i^2\right) \frac{1}{2^n}\theta^{3n} \exp\{-s_n\theta\}$$

and a conjugate prior is

$$\pi_0(\theta) \equiv \Gamma(\alpha_0, \beta_0)$$

so that the posterior is $\pi_n(\theta) \equiv Gamma(\alpha_n, \beta_n)$ where

$$\alpha_n = 3n + \alpha_0 \qquad \beta_n = s_n + \beta_0.$$

6 Marks