In this MidTerm there are three questions:

- Q1: 20 Marks
- Q2: 20 Marks
- Q3: 10 Marks

Q1. Consider realization $Y_1, \ldots, Y_n$ from an infinitely exchangeable binary sequence where, in de Finetti's representation, prior distribution π_0 on the unknown parameter θ is specified. Suppose that

$$S_n = \sum_{i=1}^{n} Y_i.$$

Let $y_1, \ldots, y_n$ and s_n be the corresponding observed values. As usual, the posterior distribution will be denoted π_n.

(a) Find $\pi_n(\theta)$ if π_0 is the $Uniform(0, 1)$ distribution. 6 Marks

(b) Find $\pi_n(\theta)$ if π_0 places equal probability on the values $1/3$ and $2/3$ and places zero probability elsewhere. 6 Marks

(c) Suppose that only the value of s_n is recorded. Is it possible to compute π_n given only this information ? Justify your answer. 4 Marks

(d) Find the Bayesian estimate of θ, $\widehat{\theta}_B$, if π_0 is the $Beta(10, 3)$ distribution, and the loss function

$$L(t, \theta) = \frac{(t-\theta)^2}{\theta(1-\theta)}$$

is used. Recall that

$$\widehat{\theta}_B = \arg\min_t \int L(t, \theta)\pi_n(\theta)\, d\theta.$$

4 Marks

Q2. A Bayesian analysis based on the conditional model

$$f_Y(y;\theta) = \frac{e^{-\theta}\theta^y}{y!} \qquad y = 0, 1, 2, \ldots$$

and zero otherwise, for $\theta > 0$, is to be carried out based on a random sample of size n.

(a) Find a conjugate prior for this problem. 4 Marks

(b) Find the Jeffreys prior for this problem. 6 Marks

(c) Find a large sample approximation to the posterior distribution π_n. 6 Marks

(d) If

$$\pi_0(\theta) = e^{-\theta} \qquad \theta > 0$$

find the posterior distribution for the data set

$$3, 0, 2, 2, 4, 2$$

which represent a sample of size $n = 6$. 4 Marks

Q3. A Bayesian analysis based on the conditional model

$$f_Y(y;\theta) \equiv Normal(\theta, 4)$$

is to be carried out based on a random sample of size n. The conjugate prior

$$\pi_0(\theta) \equiv Normal(\eta_0, 1/\lambda_0)$$

is to be used, for hyperparameters η_0, λ_0.

(a) Find the posterior distribution for θ using the complete-the-square identity

$$A(x-a)^2 + B(x-b)^2 = (A+B)(x-m)^2 + \frac{AB}{A+B}(a-b)^2$$

where

$$m = \frac{Aa+Bb}{A+B}.$$

6 Marks

(b) The Bayes estimate for θ under quadratic loss is the posterior mean. Under what (limiting) condition does the Bayes estimate equal the maximum likelihood estimate ? Justify your answer. 4 Marks