MATH 557 - MID-TERM 2017 - SOLUTIONS

1. (a) We have

$$f_{\mathbf{X}}(\mathbf{x};\alpha,\beta) = \left\{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right\}^n \left\{\prod_{i=1}^n x_i\right\}^{\alpha-1} \left\{\prod_{i=1}^n (1-x_i)\right\}^{\beta-1}$$

suggesting the sufficient statistic $\mathbf{T}(\mathbf{X}) = (\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i))^\top$ and the result follows using the Fisher-Neyman Factorization Theorem. 3 MARKS

(b) Writing $\lambda = \log\theta$, we realize that this is the $Poisson(\log\theta)$ model. Hence by elementary calculation, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a sufficient statistic for $\log\theta$. 3 MARKS

(c) The joint pdf is only non-zero if $X_i > \theta$ for all i, and hence can be written

$$f_{\mathbf{X}}(\mathbf{x};\theta) \quad = \quad \frac{\mathbb{1}_{(x_{(1)},\infty)}(\theta)}{\theta^n} \exp\left\{-\frac{1}{\theta}\sum_{i=1}^n x_i - n\right\}$$

and it follows that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, \sum_{i=1}^n X_i)$ is a sufficient statistic. 4 MARKS

2. (a) Note first that by standard expansion into a quartic polynomial

$$\left(\frac{x-\theta}{\sigma}\right)^4 = w_0(\theta,\sigma) + \sum_{j=1}^4 w_j(\theta,\sigma)x^j = w_0(\theta,\sigma) + \sum_{j=1}^4 w_j(\theta,\sigma)t_j(x)$$

say, where $w_j(\theta,\sigma)$ are constant functions of θ and σ. Thus

$$f_X(x;\theta,\sigma) = h(x)c(\theta,\sigma)\exp\left\{\sum_{j=1}^k w_j(\theta,\sigma)t_j(x)\right\}$$

where $h(x) = 1$, $c(\theta,\sigma) = \exp\{w_0(\theta,\sigma) - \kappa(\theta,\sigma)\}$, $t_j(x) = x^j, j = 1,\ldots,4$, and hence the distribution is an Exponential Family distribution. By inspection, and using the Neyman factorization theorem in this Exponential family setting, we have

$$\mathbf{T}(\mathbf{X}) = (T_1(\mathbf{X}), T_2(\mathbf{X}), T_3(\mathbf{X}), T_4(\mathbf{X}))^\top \qquad T_j(\mathbf{X}) = \sum_{i=1}^n t_j(X_i) = \sum_{i=1}^n X_i^j \qquad j = 1,\ldots,4$$

is a sufficient statistic. As this is a regular Exponential Family distribution, it follows that this statistic is also minimal sufficient; this is easily verified using the minimal sufficiency theorem, as the log density is a polynomial function. 6 MARKS

(b) This model is also a location family with standard member

$$f_0(x) = c\exp\{-x^4\} \qquad x\in\mathbb{R}.$$

Hence we may write for $i = 1,\ldots,n$, $X_i \stackrel{d}{=} Z_i + \theta$, where $Z_i \sim f_0$. Consider the minimum and maximum order statistics $X_{(1)}$ and $X_{(n)}$, and range $R = X_{(n)} - X_{(1)}$. As

$$R = X_{(n)} - X_{(1)} \stackrel{d}{=} Z_{(n)} - Z_{(1)},$$

it follows that R is ancillary, as its distribution does not depend on θ. 4 MARKS

3. (a) The likelihood is

$$\mathscr{L}(\mathbf{x};\theta) = \left\{\prod_{i=1}^{n} \mathbb{1}_{(0,1)}(x_i)\right\} \theta^n \left\{\prod_{i=1}^{n} (1-x_i)\right\}^{\theta-1} = h(\mathbf{x})\theta^n\{T(\mathbf{x})\}^{\theta-1} \propto \theta^n\{T(\mathbf{x})\}^{\theta}$$

say, where $T(\mathbf{x}) = \prod_{i=1}^{n}(1-x_i)$. The log-likelihood is therefore

$$\ell(\mathbf{x};\theta) = \text{const.} + n\log\theta + \theta\log T(\mathbf{x})$$

with derivative

$$\dot{\ell}(\mathbf{x};\theta) = \frac{n}{\theta} + \log T(\mathbf{x})$$

and this equating to zero we find that the MLE is

$$\widehat{\theta}_n = -\frac{n}{\log T(\mathbf{x})} = -\frac{n}{\sum\limits_{i=1}^{n}\log(1-x_i)}$$

It is easy to check that the second derivative is negative at this solution, taking the value

$$-\frac{n}{\widehat{\theta}_n^2} < 0.$$

5 MARKS

(b) The likelihood is

$$\mathscr{L}(\mathbf{x};\alpha,\beta) = \left\{\prod_{i=1}^{n} \mathbb{1}_{(0,\beta)}(x_i)\right\} \frac{\alpha^n}{\beta^{n\alpha}} \left\{\prod_{i=1}^{n} x_i\right\}^{\alpha-1}$$

Let $T(\mathbf{x}) = \prod_{i=1}^{n} x_i$, and note that

$$\prod_{i=1}^{n} \mathbb{1}_{(0,\beta)}(x_i) \equiv \mathbb{1}_{(0,\beta)}(x_{(n)})$$

The log-likelihood is therefore

$$\ell(\mathbf{x};\alpha,\beta) = \begin{cases} n\log\alpha - n\alpha\log\beta + (\alpha-1)\log T(\mathbf{x}) & \beta > x_{(n)} \\ -\infty & \beta \le x_{(n)} \end{cases}$$

It is evident that as the parameter space dictates that $\alpha > 0$, this log-likelihood is monotonic decreasing in β for $\beta > x_{(n)}$ (and equal to negative infinity on $(0, x_{(n)})$), so therefore the MLE for β must be $x_{(n)}$. For α, the partial derivative is

$$\frac{\partial \ell(\mathbf{x};\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} - n\log\beta + \log T(\mathbf{x})$$

so therefore equating to zero and solving at $\beta = \widehat{\beta}_n = x_{(n)}$, we have

$$\widehat{\alpha}_n = -\frac{n}{n\log\widehat{\beta}_n - \log T(\mathbf{x})} = \frac{n}{\sum\limits_{i=1}^{n}(\log x_{(n)} - \log x_i)}$$

At this solution, the second derivative is $-n/\widehat{\alpha}_n^2 < 0$.

5 MARKS

4. (a) It is useful to re-write this density as

$$f_X(x;\theta_1,\theta_2) = \frac{1}{\theta_1+\theta_2}\left\{\exp\left\{\frac{x}{\theta_2}\right\}\right\}^{\mathbb{1}_{(-\infty,0]}(x)}\left\{\exp\left\{-\frac{x}{\theta_1}\right\}\right\}^{\mathbb{1}_{(0,\infty)}(x)}$$
$$= \frac{1}{\theta_1+\theta_2}\exp\left\{\frac{x\mathbb{1}_{(-\infty,0]}(x)}{\theta_2}\right\}\exp\left\{-\frac{x\mathbb{1}_{(0,\infty)}(x)}{\theta_1}\right\}$$
$$= \frac{1}{\theta_1+\theta_2}\exp\left\{-\frac{-x\mathbb{1}_{(-\infty,0]}(x)}{\theta_2}-\frac{x\mathbb{1}_{(0,\infty)}(x)}{\theta_1}\right\}$$

and hence the likelihood can be written

$$\mathscr{L}(\mathbf{x};\theta_1,\theta_2) = \left(\frac{1}{\theta_1+\theta_2}\right)^n \exp\left\{-\frac{T_2}{\theta_2}-\frac{T_1}{\theta_1}\right\}$$

for the statistics

$$T_1 = \sum_{i=1}^n \mathbb{1}_{(0,\infty)}(x_i)x_i \qquad\qquad T_2 = -\sum_{i=1}^n \mathbb{1}_{(-\infty,0]}(x_i)x_i$$

and hence the MLEs must be functions of these sufficient statistics as required.

6 MARKS

(b) For a sample of size $n=1$, we have that

$$\frac{\partial^2\theta}{\partial\theta\partial\theta^\top}\{\log f_X(X;\theta)\} = \begin{bmatrix} \dfrac{1}{(\theta_1+\theta_2)^2}-\dfrac{2T_1}{\theta_1^3} & \dfrac{1}{(\theta_1+\theta_2)^2} \\ \dfrac{1}{(\theta_1+\theta_2)^2} & \dfrac{1}{(\theta_1+\theta_2)^2}-\dfrac{2T_2}{\theta_2^3} \end{bmatrix}$$

Now, by direct calculation

$$\mathbb{E}_{T_1}[T_1;\theta_1,\theta_2] = \int_{-\infty}^{\infty}\mathbb{1}_{(0,\infty)}(x)xf_X(x;\theta_1,\theta_2)\,dx = \int_0^\infty x\frac{1}{(\theta_1+\theta_2)}\exp\{-x/\theta_1\}\,dx = \frac{\theta_1^2}{(\theta_1+\theta_2)}$$

and similarly

$$\mathbb{E}_{T_2}[T_2;\theta_1,\theta_2] = \frac{\theta_2^2}{(\theta_1+\theta_2)}$$

and hence

$$\mathcal{I}_\theta(\theta) = \begin{bmatrix} -\dfrac{1}{(\theta_1+\theta_2)^2}+\dfrac{2}{\theta_1(\theta_1+\theta_2)} & -\dfrac{1}{(\theta_1+\theta_2)^2} \\ -\dfrac{1}{(\theta_1+\theta_2)^2} & \dfrac{2}{\theta_2(\theta_1+\theta_2)}. \end{bmatrix}$$

$$= \frac{1}{(\theta_1+\theta_2)^2}\begin{bmatrix} 1+\dfrac{2\theta_2}{\theta_1} & -1 \\ -1 & 1+\dfrac{2\theta_1}{\theta_2} \end{bmatrix}.$$

Evaluating at $\theta=\theta_0$ gives the result.

4 MARKS