MATH 557 - EXERCISES 2
SOLUTIONS

1 The likelihood in the original parameterization is

$$\mathscr{L}_n(\theta_1, \theta_2 | x_1, x_2) = \binom{n_1}{x_1} \theta_1^{x_1} (1-\theta_1)^{n_1 - x_1} \binom{n_2}{x_2} \theta_2^{x_2} (1-\theta_2)^{n_2 - x_2}$$

If $\phi = \theta_2/(1-\theta_2)$, then

$$\theta_2 = \phi/(1+\phi) \qquad\qquad \theta_1 = (\phi\psi)/(1+\phi\psi).$$

Either by writing out the likelihood in full in the new parameterization, and maximizing in the usual way, or by using invariance properties of maximum likelihood estimates, we conclude that

$$\widehat{\psi}(x_1, x_2) = \frac{\widehat{\theta}_1/(1-\widehat{\theta}_1)}{\widehat{\theta}_2/(1-\widehat{\theta}_2)} = \frac{(x_1/n_1)/(1-x_1/n_1)}{(x_2/n_2)/(1-x_2/n_2)} = \frac{x_1/(n_1 - x_1)}{x_2/(n_2 - x_2)}$$

2 $X_1, ..., X_n \sim Gamma(\alpha, \beta)$ so that $\mathbb{E}_{Xi}[X_i; \alpha, \beta] = \alpha/\beta$ and $\text{Var}_{X_i}[X_i; \alpha, \beta] = \alpha/\beta^2$ so that

$$\mathbb{E}_{X_i}\left[X_i^2; \alpha, \beta\right] = \text{Var}_{X_i}[X_i; \alpha, \beta] + \{\mathbb{E}_{X_i}[X_i; \alpha, \beta]\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha+1)}{\beta^2}$$

Hence for the method of moments estimators $\widehat{\alpha}_{MM}$ and $\widehat{\beta}_{MM}$, need to solve the following:

FIRST MOMENT	Solve	$\frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x} = \frac{\alpha}{\beta}$
SECOND MOMENT	Solve	$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = (\overline{x})^2 + S^2 = \frac{\alpha(\alpha+1)}{\beta^2}$

where $S^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$. Elementary algebra gives $\widehat{\alpha}_{MM} = \frac{(\overline{x})^2}{S^2}$ and $\widehat{\beta}_{MM} = \frac{\overline{x}}{S^2}$.

3 Writing $\mathscr{L}_n(\theta)$ for $\mathscr{L}(\mathbf{x}; \theta)$ throughout:

(i) For $\theta > 0$

STEP 1 $$\mathscr{L}_n(\theta) = \prod_{i=1}^{n} f_X(x_i; \theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$$

STEP 2 $$\log \mathscr{L}_n(\theta) = n \log\theta + (\theta - 1)\sum_{i=1}^{n} \log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log \mathscr{L}_n(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i = 0 \qquad \Longrightarrow \qquad \widehat{\theta}_{ML} = -n/\sum_{i=1}^{n} \log x_i$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log \mathscr{L}_n(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \widehat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^{n} \log x_i} \qquad\qquad \text{ESTIMATOR}: \ -\frac{n}{\sum_{i=1}^{n} \log X_i}$$

(ii) For $\theta > 0$

STEP 1 $$\mathscr{L}_n(\theta) = \prod_{i=1}^n f_X(x_i;\theta) = \prod_{i=1}^n (\theta+1)\, x_i^{-(\theta+2)} = (\theta+1)^n \left(\prod_{i=1}^n x_i\right)^{-(\theta+2)}$$

STEP 2 $$\log \mathscr{L}_n(\theta) = n\log(\theta+1) - (\theta+2)\sum_{i=1}^n \log x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log \mathscr{L}_n(\theta)\} = \frac{n}{\theta+1} - \sum_{i=1}^n \log x_i = 0 \qquad \Longrightarrow \qquad \widehat{\theta}_{ML} = n/\sum_{i=1}^n \log x_i - 1$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log \mathscr{L}_n(\theta)\} = -\frac{n}{(\theta+1)^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \widehat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^n \log x_i} - 1 \qquad \text{ESTIMATOR:} = \frac{n}{\sum\limits_{i=1}^n \log x_i} - 1$$

(iii) For $\theta > 0$

STEP 1 $$\mathscr{L}_n(\theta) = \prod_{i=1}^n f_X(x_i;\theta) = \prod_{i=1}^n \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n}\left(\prod_{i=1}^n x_i\right)\exp\left\{-\theta\sum_{i=1}^n x_i\right\}$$

STEP 2 $$\log \mathscr{L}_n(\theta) = 2n\log\theta + \sum_{i=1}^n \log x_i - \theta\sum_{i=1}^n x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log \mathscr{L}_n(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^n x_i = 0 \qquad \Longrightarrow \qquad \widehat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^n x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log \mathscr{L}_n(\theta)\} = -\frac{2n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE}: \widehat{\theta}_{ML} = \frac{2n}{\sum\limits_{i=1}^n x_i} \qquad \text{ESTIMATOR:} = \frac{2n}{\sum\limits_{i=1}^n X_i}$$

(iv) Because of the constraint in the pdf that $x \leq \theta$

STEP 1 $$\mathscr{L}_n(\theta) \quad \prod_{i=1}^n f_X(x_i;\theta) == \begin{cases} \prod\limits_{i=1}^n 2\theta^2 x_i^{-3} = 2^n\theta^{2n}\left(\prod\limits_{i=1}^n x_i^{-3}\right) & \theta \leq x_1, ..., x_n \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log \mathscr{L}_n(\theta) = n\log 2 + 2n\log\theta - 3\sum_{i=1}^n \log x_i$$

At this point we note that the likelihood is monotonically increasing in θ, and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \leq x_1, ..., x_n$ is still satisfied, hence

$$\text{ESTIMATE}: \widehat{\theta}_{ML} = \min\{x_1, ..., x_n\} \qquad \text{ESTIMATOR: } \min\{X_1, ..., X_n\}$$

(v) For $\theta > 0$

STEP 1 $$\mathscr{L}_n(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \left(\frac{\theta}{2}\right) \exp\{-\theta\,|x_i|\} = 2^{-n}\theta^n \exp\left\{-\theta \sum_{i=1}^{n} |x_i|\right\}$$

STEP 2 $$\log \mathscr{L}_n(\theta) = -n\log 2 + n\log\theta - \theta\sum_{i=1}^{n}|x_i|$$

STEP 3 $$\frac{d}{d\theta}\{\log \mathscr{L}_n(\theta)\} = \frac{n}{\theta} - \sum_{i=1}^{n}|x_i| = 0 \qquad \Longrightarrow \qquad \widehat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^{n}|x_i|}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log \mathscr{L}_n(\theta)\} = -\frac{n}{\theta^2} < 0 \qquad \text{for all } \theta$$

Hence

$$\text{ESTIMATE} : \widehat{\theta}_{ML} = \frac{n}{\sum\limits_{i=1}^{n}|x_i|} \qquad\qquad \text{ESTIMATOR} : \frac{n}{\sum\limits_{i=1}^{n}|x_i|}$$

(vi) Because of the constraint in the pdf that $\theta_1 \le x \le \theta_2$

STEP 1 $$\mathscr{L}_n(\theta_1,\theta_2) \quad \prod_{i=1}^{n} f_X(x_i;\theta) = \begin{cases} \prod\limits_{i=1}^{n} \dfrac{1}{(\theta_2-\theta_1)} = \dfrac{1}{(\theta_2-\theta_1)^n} & \theta_1 \le x_1,\dots,x_n \le \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

STEP 2 $$\log \mathscr{L}_n(\theta_1,\theta_2) = -n\log(\theta_2-\theta_1)$$

At this point we note that the likelihood is monotonically increasing in $\theta_1$and monotonically decreasing in θ_2, and hence the likelihood is maximized when θ_1 is as **large** as possible and when θ_2 is as **small** as possible, but so that the constraint $\theta_1 \le x_1,\dots,x_n \le \theta_2$ (and $\theta_2 \ge \theta_1$) is still satisfied, hence

$$\text{ESTIMATE} : \begin{aligned} \widehat{\theta}_{1ML} &= \min\{x_1,\dots,x_n\} \\ \widehat{\theta}_{2ML} &= \max\{x_1,\dots,x_n\} \end{aligned} \qquad\qquad \text{ESTIMATOR:} \begin{aligned} \theta_1 &\quad \min\{X_1,\dots,X_n\} \\ \theta_2 &\quad \max\{X_1,\dots,X_n\} \end{aligned}$$

(vii) Noting the constraint in the pdf that $x \ge \theta_2$, we have

STEP 1 $$\mathscr{L}_n(\theta_1,\theta_2) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \theta_1\theta_2^{\theta_1} x_i^{-(\theta_1+1)} = \theta_1^n \theta_2^{n\theta_1}\left(\prod_{i=1}^{n} x_i\right)^{-(\theta_1+1)}$$
$$0 \le x_1,\dots,x_n \le \theta_2$$

STEP 2 $$\log \mathscr{L}_n(\theta_1,\theta_2) = n\log\theta_1 + n\theta_1\log\theta_2 - (\theta_1+1)\sum_{i=1}^{n}\log x_i$$

STEP 3 $$\frac{\partial}{\partial\theta_1}\{\log \mathscr{L}_n(\theta_1,\theta_2)\} = \frac{n}{\theta_1} + n\log\theta_2 - \sum_{i=1}^{n}\log x_i = 0$$
$$\Longrightarrow \quad \widehat{\theta}_{1ML} = \frac{n}{\sum\limits_{i=1}^{n}\log x_i - n\log\widehat{\theta}_{2ML}}$$
$$\frac{\partial}{\partial\theta_2}\{\log \mathscr{L}_n(\theta_1,\theta_2)\} = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when θ_2 is as **large** as possible, that is, when $\widehat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$ .Hence

ESTIMATES

$$\widehat{\theta}_{1ML} = \frac{n}{\left[\sum\limits_{i=1}^{n} \log x_i - n \log\{\min\{x_1, ..., x_n\}\}\right]}$$

$$\widehat{\theta}_{2ML} = \min\{x_1, ..., x_n\}$$

ESTIMATORS:

$$\theta_1 \qquad \frac{n}{\left[\sum\limits_{i=1}^{n} \log X_i - n \log\{\min\{X_1, ..., X_n\}\}\right]}$$

$$\theta_2 \qquad \min\{X_1, ..., X_n\}$$

4 For the $Poisson(\lambda)$ case

STEP 1 $$\mathscr{L}(\lambda) = \prod_{i=1}^{n} f_X(x_i|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum\limits_{i=1}^{n} x_i}}{\left(\prod\limits_{i=1}^{n} x_i!\right)}$$

STEP 2 $$\log\mathscr{L}(\lambda) = -\sum_{i=1}^{n} \log x_i! - n\lambda + \left(\sum_{i=1}^{n} x_i\right)\log\lambda$$

STEP 3 $$\frac{d}{d\lambda}\{\log\mathscr{L}(\lambda)\} = -n + \left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda} = 0 \qquad \Longrightarrow \qquad \widehat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$$

STEP 4 $$\frac{d^2}{d\lambda^2}\{\log\mathscr{L}(\lambda)\} = -\left(\sum_{i=1}^{n} x_i\right)\frac{1}{\lambda^2} < 0 \qquad \text{for all } \lambda$$

Therefore

ESTIMATE : $\widehat{\lambda}_{ML} = \overline{x}$ ESTIMATOR: $\overline{X}$

We see that

$$T_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \Longrightarrow \qquad \mathbb{E}_{T_1}[T_1; \lambda] = \lambda$$

and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \qquad \text{so that} \qquad \lambda = -\log\tau$$

and we can reformulate the likelihood in terms of τ, giving

$$\log\mathscr{L}(\tau) = -\sum_{i=1}^{n} \log x_i! + n\log\tau + \left(\sum_{i=1}^{n} x_i\right)\log(-\log\tau)$$

Differentiating wrt τ in the usual way and equating to zero, we find that $\widehat{\tau}_{ML} = \exp\{-\overline{x}\}$, so that $\widehat{\tau}_{ML}(\lambda) = \tau(\widehat{\lambda}_{ML})$. Using a Taylor approximation

$$g(\widehat{\lambda}) \simeq g(\lambda) + \dot{g}(\lambda)(\widehat{\lambda} - \lambda) + \ddot{g}(\lambda)(\widehat{\lambda} - \lambda)/2$$

with $g(t) = e^{-t}$ and $\widehat{\lambda} = \overline{x}$, and taking expectations, noting that $\mathbb{E}_X[\overline{X}] = \lambda$,

$$\mathbb{E}_{T_1}[\exp\{-\overline{X}\};\lambda] \approx e^{-\lambda} + e^{-\lambda}\text{Var}_{T_1}[\overline{X};\lambda]/n = e^{-\lambda} + \frac{1}{2}e^{-\lambda}\lambda/n$$

so the bias is approximately $\frac{1}{2}e^{-\lambda}\lambda/n$.

5 Noting the constraint in the pdf that $x \geq \eta$, we have

STEP 1 $\quad \mathscr{L}(\lambda,\eta) = \prod_{i=1}^{n} f_X(x_i;\theta) = \prod_{i=1}^{n} \lambda\exp\{-\lambda(x_i-\eta)\} = \lambda^n \exp\left\{-\lambda\sum_{i=1}^{n}(x_i-\eta)\right\}$

$\eta \leq x_1,...,x_n$, zero otherwise

STEP 2 $\quad \log\mathscr{L}(\lambda,\eta) = n\log\lambda - \lambda\sum_{i=1}^{n}(x_i-\eta) = n\log\lambda - \lambda\sum_{i=1}^{n}x_i + n\lambda\eta$

STEP 3 $\quad \dfrac{\partial}{\partial\lambda}\{\log\mathscr{L}(\lambda,\eta)\} = \dfrac{n}{\lambda} + \sum_{i=1}^{n}x_i - n\eta = 0 \quad \Longrightarrow \quad \widehat{\lambda}_{ML} = \dfrac{n}{\sum_{i=1}^{n}x_i - n\widehat{\eta}}$

$$\frac{\partial}{\partial\eta}\{\log\mathscr{L}(\lambda,\eta)\} = n\lambda$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when η is as **large** as possible, that is, when $\widehat{\eta} = \min\{x_1,...,x_n\}$ .Hence

ESTIMATES

$$\widehat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n}x_i - n\min\{x_1,...,x_n\}} = \frac{n}{\sum_{i=1}^{n}(x_i - \min\{x_1,...,x_n\})}$$

$$\widehat{\eta}_{ML} = \min\{x_1,...,x_n\}$$

with the obvious corresponding estimators.

6 $X_1,...,X_n \sim Exponential(1/\theta)$ so that $\mathbb{E}_{X_i}[X_i;\theta] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^{n} X_i \sim Gamma(n, 1/\theta) \qquad \Longrightarrow \qquad E_{f_X}[X] = n/(1/\theta) = n\theta$$

so that if $T_1 = \overline{X} = \dfrac{1}{n}X$ then

$$\mathbb{E}_{T_1}[T_1;\theta] = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1,...,X_n\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta} \qquad y > 0$$

so that $Y_1 \sim Exponential(n/\theta)$. Hence if $Z = nY_1$ then

$$\mathbb{E}_{Y_1}[Y_1;\theta] = \frac{\theta}{n} \qquad \Longrightarrow \qquad \mathbb{E}_Z[Z;\theta] = n\frac{\theta}{n} = \theta$$

and hence Z is an unbiased estimator of θ.

7 We have

$$f_X(x) = \frac{1}{2} \qquad \theta - 1 \le x \le \theta + 1 \qquad\qquad F_X(x) = \frac{x-(\theta-1)}{2} = \frac{x-\theta+1}{2} \qquad \theta - 1 \le x \le \theta + 1$$

$E_{X_i}[X_i;\theta] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \overline{X}$

$$\mathbb{E}_{T_1}[T_1;\theta] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{X_i}[X_i;\theta] = \frac{1}{n}\sum_{i=1}^{n}\theta = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ.

Now if $Y_1 = \min\{X_1, ..., X_n\}$and $Y_n = \max\{X_1, ..., X_n\}$, results on order statistics results give that

$$f_{Y_1}(y) = nf_X(y)\{1 - F_X(y)\}^{n-1} = n\frac{1}{2}\left\{1 - \frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1} \qquad \theta - 1 \le y \le \theta + 1$$

and

$$f_{Y_n}(y) = nf_X(y)\{F_X(y)\}^{n-1} = n\frac{1}{2}\left\{\frac{y-(\theta-1)}{2}\right\}^{n-1} = \frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} \qquad \theta - 1 \le y \le \theta + 1$$

For the expectations,

$$\begin{aligned}
\mathbb{E}_{Y_1}[Y_1;\theta] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1+\theta-y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 ((1+\theta) - 2t)\, t^{n-1}\, 2\, dt \qquad \text{setting } t = (1+\theta-y)/2 \Longrightarrow y = (1+\theta) - 2t \\
&= (1+\theta)\int_0^1 nt^{n-1}\, dt - 2n\int_0^1 t^n dt = (1+\theta) - \frac{2n}{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{Y_n}[Y_n;\theta] &= \int_{\theta-1}^{\theta+1} y\frac{n}{2}\left\{\frac{1-\theta+y}{2}\right\}^{n-1} dy \\
&= \frac{n}{2}\int_0^1 (2t - (1-\theta))\, t^{n-1}\, 2\, dt \qquad \text{setting } t = (1-\theta+y)/2 \Longrightarrow y = 2t - (1-\theta) \\
&= 2\int_0^1 nt^n\, dt - (1-\theta)\int_0^1 nt^{n-1} dt = \frac{2n}{n+1} - (1-\theta)
\end{aligned}$$

so that if $M = (Y_1 + Y_n)/2$ then by properties of expectations

$$\mathbb{E}_M[M;\theta] = \frac{1}{2}\mathbb{E}_{Y_1}[Y_1] + \frac{1}{2}\mathbb{E}_{Y_n}[Y_n] = \left[\frac{1}{2}(1+\theta) - \frac{n}{n+1}\right] + \left[\frac{n}{n+1} - \frac{1}{2}(1-\theta)\right] = \theta$$

and hence M is an unbiased estimator for θ.

8 (i) For $\lambda > 0$

STEP 1 $$\mathscr{L}(\lambda) = \prod_{i=1}^{n} f_X(x_i|\lambda) = \prod_{i=1}^{n} \frac{\lambda^2}{\Gamma(2)} x_i \exp\{-\lambda x_i\} = \frac{\lambda^{2n}}{\{\Gamma(2)\}^n} \left(\prod_{i=1}^{n} x_i\right) \exp\left\{-\lambda \sum_{i=1}^{n} x_i\right\}$$

STEP 2 $$\log \mathscr{L}(\lambda) = 2n \log \lambda - n \log \Gamma(2) + \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i$$

STEP 3 $$\frac{d}{d\theta}\{\log \mathscr{L}(\lambda)\} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \qquad \Longrightarrow \qquad \widehat{\lambda}_{ML} = \frac{2n}{\sum\limits_{i=1}^{n} x_i}$$

STEP 4 $$\frac{d^2}{d\theta^2}\{\log \mathscr{L}(\lambda)\} = -\frac{2n}{\lambda^2} < 0 \qquad \text{at } \lambda = \widehat{\lambda}_{ML}$$

Hence the estimator is

$$\widehat{\lambda} = \frac{2n}{\left(\sum\limits_{i=1}^{n} X_i\right)}$$

(ii) By the invariance property of maximum likelihood estimators, we must have that the ML estimator of $\tau = 1/\lambda$ is $T = \sum\limits_{i=1}^{n} X_i/2n$. Now, using mgfs, it is straightforward to show that

$$\sum_{i=1}^{n} X_i \sim Gamma(2n, \lambda) \qquad \Longrightarrow \qquad T = \frac{1}{2n}\sum_{i=1}^{n} X_i \sim Gamma(2n, 2n\lambda)$$

so that

$$\mathbb{E}_T[T;\lambda] = \frac{2n}{2n\lambda} = \frac{1}{\lambda} \qquad \mathrm{Var}_{f_T}[T;\lambda] = \frac{2n}{(2n\lambda)^2} = \frac{1}{2n\lambda^2} \qquad \mathbb{E}_T\left[T^2;\lambda\right] = \frac{1}{2n\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2n+1}{2n\lambda^2}$$

9 (a) We have for the likelihood

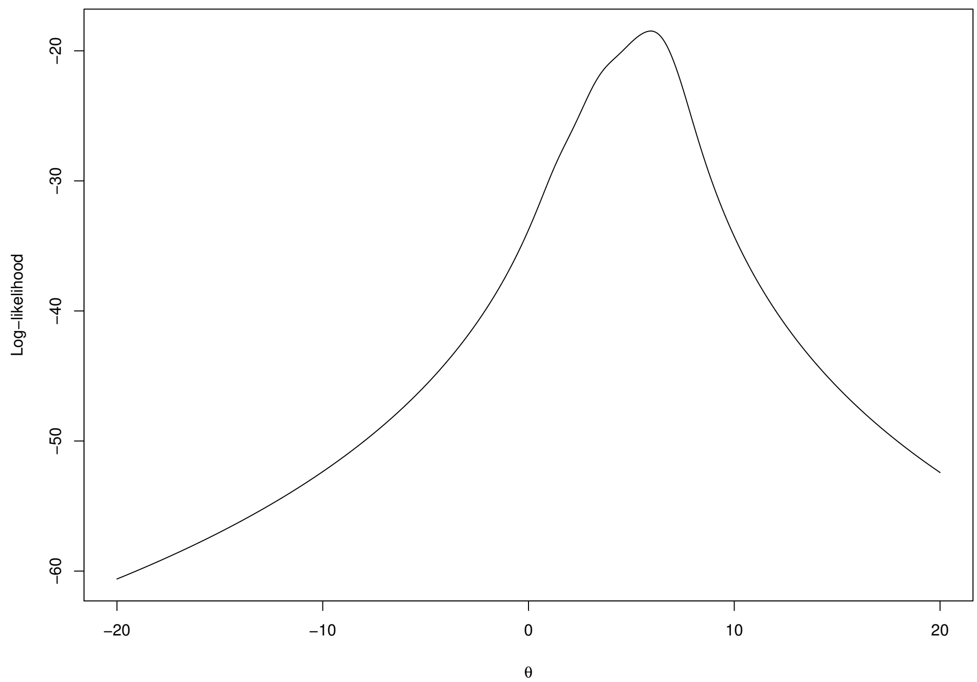
$$\begin{aligned} \mathscr{L}(\mathbf{x};\theta) &= \prod_{i=1}^{n} \frac{1}{\pi} \frac{1}{1+(x_i-\theta)^2} = \frac{1}{\pi^n} \prod_{i=1}^{n} \frac{1}{1+(x_i-\theta)^2} \\ \ell(\mathbf{x};\theta) &= -n \log \pi - \sum_{i=1}^{n} \log(1+(x_i-\theta)^2). \end{aligned}$$

yielding the following score equation (which cannot be solved analytically)

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n} \frac{2(x_i-\theta)}{1+(x_i-\theta)^2} = 0$$

(b) The code below evaluates the log-likelihood in `R` on a fine grid on the range (-20,20):

```
x.dat<-c(7.36,5.14,3.71,3.15,6.00,6.38,1.34,6.73);thvec<-seq(-20,20,length=20001)
log.pdf.func<-function(x,th){return(-log(pi)-log(1+(x-th)^2))}
log.pdf.mat<-outer(thvec,x.dat,log.pdf.func)
log.like.vec<-apply(log.pdf.mat,1,sum)
```




```
plot(thvec,log.like.vec,type=''l'',xlab=expression(theta),ylab=''Log-likelihood'')
th.max<-thvec[which.max(log.like.vec)]
```

In this code, the variable `th.max` contains the numerical value of the estimate of θ; to two decimal places, we find that $\widehat{\theta}(\mathbf{x}) = 5.94$. The code also produces the following plot. The following code uses the `optimize` function to produce the same result: the commands

```
log.like.fx<-function(x,xd){return(sum(log.pdf.func(xd,x[1])))}
optimize(f=log.like.fx,xd=x.dat,interval=c(-20,20),maximum=TRUE)
```

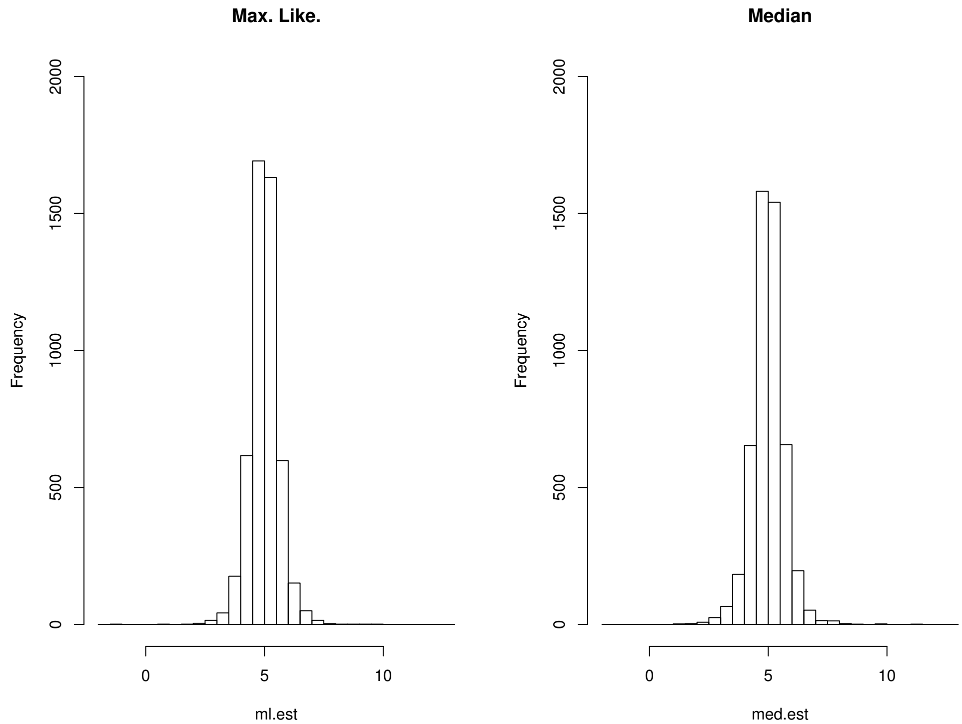
yield

```
$maximum [1] 5.93949
$objective [1] -18.48059
```

10 The following R code computes the $N = 5000$ estimates.

```
N<-5000
n<-8
ml.est<-med.est<-rep(0,N)
theta0<-5
for(i in 1:N){
    x<-rcauchy(n)+theta0
    ml<-optimize(f=log.like.fx,xd=x,interval=c(-20,20),maximum=TRUE)
    ml.est[i]<-ml$maximum
    med.est[i]<-median(x)
}
```

The histograms below depict the 5000 maximum likelihood and median estimates:



It can be discerned from these figures that the variance of the median estimator is slightly larger than for the maximum likelihood estimator; in one simulation, I got that the variance for the sample of median estimates was 0.468, and the variance for the sample of maximum likelihood estimates was 0.387.