MATH 557 - PRACTICE MID-TERM SOLUTIONS

1. (a) Y is sufficient for θ, as, given $Y = y$, the conditional distribution of X is concentrated on $\{-y, y\}$ irrespective of the value of θ. 4 MARKS

   (b) The likelihood is

$$\mathscr{L}(\mathbf{x};\theta) = f_{\mathbf{X}}(\mathbf{x};\theta) = \frac{\prod_{i=1}^{n} \mathbb{1}_{(-\theta,2\theta)}(x_i)}{3\theta} = \frac{\mathbb{1}_{(T(\mathbf{x}),\infty)}(\theta)}{3^n\theta^n}$$

   where $T(\mathbf{x}) = \max\{-x_{(1)}, x_{(n)}/2\}$. This function is monotonic decreasing in θ, so therefore $\widehat{\theta}_{ML} = T(\mathbf{x})$. By the factorization theorem, $\widehat{\theta}_{ML}(\mathbf{X})$ is sufficient. 6 MARKS

2. (a) The family $Normal(0,\theta)$ for $0 < \theta < \infty$ is **not** complete, as if $g(t) = t$, then

$$\mathbb{E}_T[g(T);\theta] = \mathbb{E}_T[T;\theta] = 0$$

   for all θ, even though $g(T)$ is not zero with probability 1.

   4 MARKS

   (b) The joint pdf is

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x};\theta_1,\theta_2) &= \frac{\prod_{i=1}^{n} \mathbb{1}_{(\theta_1,\infty)}(x_i)}{\theta_2^n} \exp\left\{-\frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)\right\} \\ &= \frac{\prod_{i=1}^{n} \mathbb{1}_{(x_{(1)},\infty)}(\theta_1)}{\theta_2^n} \exp\left\{-\frac{\sum_{i=1}^{n} x_i}{\theta_2} - n\theta_1/\theta_2)\right\} \end{aligned}$$

   Therefore, by the minimal sufficiency theorem, as the ratio $f_{\mathbf{X}}(\mathbf{x};\theta_1,\theta_2)/f_{\mathbf{X}}(\mathbf{y};\theta_1,\theta_2)$ is independent of (θ_1,θ_2) for all possible values of the parameters if and only if

$$x_{(1)} = y_{(1)} \qquad \textbf{and} \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

   it follows that $\mathbf{T}(\mathbf{X}) = (X_{(1)}, \sum_{i=1}^{n} X_i)$ is a minimal sufficient statistic. 6 MARKS

3. (a) From lectures (must show for full marks), the ML estimator is $\overline{X}$, and as $\tau(\lambda) = e^{-2\lambda}$, by **invariance** $\widehat{\tau}_{ML}(\mathbf{X}) = \exp\{-2\overline{X}\}$. 4 MARKS

   (b) We have that

$$\mathbb{E}_{X_1}[\widehat{\tau}_1(\mathbf{X});\lambda] = \mathbb{E}_{X_1}[(-1)^{X_1};\lambda] = \sum_{x=0}^{\infty}(-1)^x\frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(-\lambda)^x}{x!} = e^{-2\lambda}$$

   4 MARKS

   (c) It is unbiased, but only takes values in the set $\{-1, 1\}$, whatever the values of λ and the data. Clearly, -1 is not sensible as an estimate of λ, and this estimator is not reasonable, despite its unbiasedness. 2 MARKS

4. (a) Consider

$$T(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,c]}(X_i).$$

Then

$$\mathbb{E}_T[T;\theta] = \frac{1}{n}\sum_{i=1}^{n} P_\theta[X_i \leq c] = \frac{n\phi(\theta)}{n} = \phi(\theta)$$

5 MARKS

(b) We have for the defined $Y_1,\ldots,Y_n$ that

$$f_{Y_i}(y;\theta) = \theta^y(1-\theta)^{1-y}$$

and hence

$$\sum_{i=1}^{n} \mathbb{1}_{(-\infty,a]}(X_i) = \sum_{i=1}^{n} Y_i \sim Binomial(n,\theta).$$

For this model, if $T_0 \equiv T_0(\mathbf{y}) = \sum\limits_{i=1}^{n} y_i$, we have $\text{Var}_{T_0}[T_0;\theta] = n\theta(1-\theta)$, and so

$$\text{Var}_T[T;\theta] = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n}.$$

5 MARKS

Note: for this model, considering the likelihood defined for the Y_i, if $\ell(.;.)$ denotes the log likelihood, we have

$$\frac{\partial \ell(\mathbf{y};\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left\{T_0\log\theta + (n-T_0)\log(1-\theta)\right\} = \frac{T_0}{\theta} - \frac{n-T_0}{1-\theta} = \frac{T_0 - n\theta}{\theta(1-\theta)}$$

The Fisher information with respect to θ, when the true value is θ_0, is

$$\mathcal{I}_{\theta_0}(\theta) = \mathbb{E}_Y\left[-\frac{d^2}{d\theta^2}\left\{\log f_Y(Y;\theta)\right\}\right] = \left(\frac{\mathbb{E}_Y[Y;\theta_0]}{\theta^2} + \frac{1-\mathbb{E}_Y[Y;\theta_0]}{(1-\theta)^2}\right) = \frac{\theta_0}{\theta^2} + \frac{1-\theta_0}{(1-\theta)^2}$$

and hence

$$\mathcal{I}_{\theta_0}(\theta_0) = \frac{1}{\theta_0(1-\theta_0)}.$$

Note also that

$$\mathbb{E}_Y\left[\left(\frac{d}{d\theta}\left\{\log f_Y(Y;\theta)\right\}\right)^2;\theta_0\right] = \mathbb{E}_Y\left[\left(\frac{Y}{\theta} - \frac{1-Y}{1-\theta}\right)^2;\theta_0\right] = \mathbb{E}_Y\left[\frac{Y}{\theta^2} + \frac{(1-Y)}{(1-\theta)^2};\theta_0\right]$$

as $Y^2 \equiv Y, (1-Y)^2 \equiv (1-Y)$ and $Y(1-Y) \equiv 0$. Hence

$$\mathbb{E}_Y\left[\left(\frac{d}{d\theta}\left\{\log f_Y(Y;\theta)\right\}\right)^2;\theta_0\right] = \frac{\theta_0}{\theta^2} + \frac{1-\theta_0}{(1-\theta)^2}$$

Hence here

$$\mathcal{I}_{\theta_0}(\theta) = \mathbb{E}_{\mathbf{Y}}\left[-\frac{d^2}{d\theta^2}\left\{\log f_Y(Y;\theta)\right\}\right] = \mathbb{E}_Y\left[\left(\frac{d}{d\theta}\left\{\log f_Y(Y;\theta)\right\}\right)^2;\theta_0\right].$$