MATH 557 - EXERCISES 3
SOLUTIONS

1. Let $X \sim Binomial(n, \theta)$ for $0 < \theta < 1$.

   (a) By standard results for expectations

$$\mathbb{E}_T[T;\theta] = \frac{1}{n}\mathbb{E}_X[X;\theta] = \frac{n\theta}{n} = \theta$$

   (b) Suppose that $T'(X)$ is an unbiased estimator of $\tau(\theta)$, so that

$$\mathbb{E}_{T'}[T';\theta] = \sum_{x=0}^{n} T'(x)\binom{n}{x}\theta^x(1-\theta)^{n-x} = \frac{1}{\theta}.$$

   Clearly $T'(x)$ must be finite on $0 \leq x \leq n$, otherwise the expectation is not finite, in which case $T'(X)$ is biased for any θ. But also, for $0 < \theta < 1$

$$\sum_{x=0}^{n} T'(x)\binom{n}{x}\theta^x(1-\theta)^{n-x} \leq \sum_{x=0}^{n} T'(x)\binom{n}{x} = M(n) < \infty$$

   so therefore the expectation of $T'(X)$ is bounded above. Therefore, if $\theta < 1/M(n)$, the expectation cannot attain $1/\theta$, so the estimator is not unbiased; no unbiased estimator exists.

2. (a) The ML estimator of θ is $M(\mathbf{X}) = X_{(n)} = \max\{X_1, \ldots, X_n\}$ which has pdf

$$f_M(m;\theta) = \frac{nm^{n-1}}{\theta^n} \qquad\qquad 0 < m < \theta$$

   and zero otherwise, which has expectation

$$\mathbb{E}_M[M;\theta] = \int_0^\theta \frac{nm^n}{\theta^n}\,dm = \frac{n}{n+1}\theta$$

   so therefore the statistic $T = (n+1)M/n$ is unbiased for θ.

   (b) The variance/MSE of $T(\mathbf{X})$ is

$$\begin{aligned}
\mathrm{Var}_T[T;\theta] &= \mathbb{E}_T[T^2;\theta] - \theta^2 = \left(\frac{n+1}{n}\right)^2 \mathbb{E}_M[M^2;\theta] - \theta^2 \\
&= \left(\frac{n+1}{n}\right)^2 \int_0^\theta m^2 \frac{nm^{n-1}}{\theta^n}\,dm - \theta^2 = \left(\frac{n+1}{n}\right)^2\left(\frac{n}{n+2}\right)\theta^2 - \theta^2
\end{aligned}$$

   (c) The Cramér-Rao bound is, in this case,

$$B(\theta) = \frac{\left(\dfrac{d}{d\theta}\left\{\mathbb{E}_T[T;\theta]\right\}\right)^2}{n\mathbb{E}_X\left[U(X;\theta)^2\right]} = \frac{(\dot{\tau}(\theta))^2}{n/\theta^2} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$$

   and so the difference is

$$\theta^2\left(\left(\frac{n+1}{n}\right)^2\left(\frac{n}{n+2}\right) - 1 - \frac{1}{n}\right)$$

3. (a) The likelihood, log-likelihood and derivative are

$$\begin{aligned}
\mathscr{L}(\mathbf{x};\theta) &= \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1} \qquad \theta > 0 \\
\ell(\mathbf{x};\theta) &= n\log\theta + (\theta-1)\sum_{i=1}^n \log x_i \\
\dot{\ell}(\mathbf{x};\theta) &= \frac{n}{\theta} + \sum_{i=1}^n \log x_i
\end{aligned}$$

and hence

$$\widehat{\theta}_n(\mathbf{X}) = -\frac{n}{\sum\limits_{i=1}^n \log X_i} = \frac{n}{T(\mathbf{X})} \quad \text{say.}$$

(b) If $X \sim \text{Beta}(\theta,1)$ then if $Y = -\log X$, we have from first principles

$$\begin{aligned}
F_Y(y;\theta) &= \Pr[Y \le y|\theta] = \Pr[-\log X \le y|\theta] = \Pr[X \ge e^{-y}|\theta] = 1 - F_{X|\theta}(e^{-y}|\theta) \\
\therefore \quad f_Y(y;\theta) &= e^{-y} f_X(e^{-y};\theta) = e^{-y}\theta(e^{-y})^{\theta-1} = \theta e^{-\theta y} \qquad y > 0
\end{aligned}$$

so $Y \sim Exponential(\theta)$ and hence $T(\mathbf{X}) \sim Gamma(n,\theta)$. Now for $r = 1,2,\ldots$

$$\mathbb{E}_T[T^{-r};\theta] = \int_0^\infty \frac{1}{t^r}\frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^n}{\Gamma(n)}\frac{\Gamma(n-r)}{\theta^{n-r}} = \frac{\theta^r}{(n-1)(n-2)\ldots(n-r+1)}.$$

Hence

$$\mathbb{E}_T[T^{-1};\theta] = \frac{\theta}{(n-1)} \qquad \mathbb{E}_T[T^{-2};\theta] = \frac{\theta^2}{(n-1)(n-2)}$$

so that

$$\text{Var}_T[T^{-1};\theta] = \frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)^2(n-2)}$$

Thus the expectation and variance of $\widehat{\theta}_n(\mathbf{X})$ are

$$\frac{n\theta}{(n-1)} \qquad \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

so $\widehat{\theta}_n(\mathbf{X})$ is **not** unbiased for θ.

(c) Writing $T_1(\mathbf{X}) = \widehat{\theta}_n(\mathbf{X})$, the Cramér-Rao bound is

$$\begin{aligned}
B(\theta) &= \frac{\left(\dfrac{d}{d\theta}\{\mathbb{E}_{T_1}[T_1;\theta]\}\right)^2}{n\mathbb{E}_X\left[U(X;\theta)^2;\theta\right]} = \frac{(n/(n-1))^2}{\mathbb{E}_T\left[(n/\theta - T)^2;\theta\right]} = \frac{\left(\dfrac{n}{n-1}\right)^2}{\dfrac{n^2}{\theta^2} - 2\dfrac{n}{\theta}\mathbb{E}_T[T;\theta] + \mathbb{E}_T[T^2;\theta]} \\
&= \frac{\left(\dfrac{n}{n-1}\right)^2}{\dfrac{n^2}{\theta^2} - 2\dfrac{n}{\theta}\dfrac{n}{\theta} + \dfrac{n(n+1)}{\theta^2}} = \frac{n\theta^2}{(n-1)^2}
\end{aligned}$$

Thus the Cramér-Rao bound is **not** met here despite the fact that this is an Exponential Family distribution. The Exponential Family does yield unbiased estimators of parameters, but only in specific parameterizations relating to the natural (or canonical) parameterization. Here

$$f_X(x;\theta) = \theta \exp\{(\theta - 1)\log x\}$$

so that, by properties of the score function $U(x;\theta)$, $\mathbb{E}_X[-\log X;\theta] = 1/\theta = \tau(\theta)$ say, and the result yields that

$$T_2(\mathbf{X}) = -\frac{1}{n}\sum_{i=1}^{n} \log X_i$$

is an unbiased estimator of $\tau(\theta)$. As

$$T_2(\mathbf{X}) \sim Gamma(n, n\theta) \qquad \therefore \qquad \text{Var}_{T_2}[T_2;\theta] = \frac{1}{n\theta^2}$$

for which the Cramér-Rao bound is

$$B(\theta) = \frac{\left(\dfrac{d}{d\theta}\{\mathbb{E}_{T_2}[T_2;\theta]\}\right)^2}{n\mathbb{E}_X\left[U(X;\theta)^2;\theta\right]} = \frac{(\dot{\tau}(\theta))^2}{\mathbb{E}_T\left[(n/\theta - nT_2)^2;\theta\right]} = \frac{1/\theta^4}{n^2\text{Var}_T[T_2;\theta]} = \frac{1}{n\theta^2}$$

so the bound **is** attained for this estimator of $\tau(\theta)$.

4. (a) We have that

$$\begin{aligned}
\mathcal{I}_\theta(\theta) = \mathbb{E}_X\left[U(X;\theta)^2;\theta\right] &= \int \left\{\frac{\partial}{\partial\theta}\log f_X(x;\theta)\right\}^2 f_X(x;\theta)\,dx \\
&= \int \left\{\frac{\dot{f}(x-\theta)}{f(x-\theta)}\right\}^2 f(x-\theta)\,dx = \int \left\{\frac{\dot{f}(x)}{f(x)}\right\}^2 f(x)\,dx
\end{aligned}$$

after making the change of variables $x \longrightarrow x - \theta$. Hence $\mathcal{I}_\theta(\theta)$ is constant in θ.

(b) For the Double Exponential model

$$\begin{aligned}
\mathcal{I}_\theta(\theta) &= \int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial\theta}\log f_X(x;\theta)\right\}^2 f_X(x;\theta)\,dx = \int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial\theta} - |x-\theta|\right\}^2 \frac{1}{2}e^{-|x-\theta|}\,dx \\
&= \int_{-\infty}^{\theta} \left\{\frac{\partial}{\partial\theta}(x-\theta)\right\}^2 \frac{1}{2}e^{(x-\theta)}\,dx + \int_{\theta}^{\infty} \left\{\frac{\partial}{\partial\theta} - (x-\theta)\right\}^2 \frac{1}{2}e^{-(x-\theta)}\,dx \\
&= \int_{-\infty}^{\theta} \frac{1}{2}e^{(x-\theta)}\,dx + \int_{\theta}^{\infty} \frac{1}{2}e^{-(x-\theta)}\,dx = 1
\end{aligned}$$

Note that the lack of differentiability of the pdf at θ does not render this computation invalid, as this is a single point in a continuous distribution and hence has probability zero, that is, the function is differentiable *almost everywhere*.

(c) For the Logistic model

$$\begin{aligned}
\mathcal{I}_\theta(\theta) &= \int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial\theta}\log f_X(x;\theta)\right\}^2 f_X(x;\theta)\,dx = \int_{-\infty}^{\infty} \left\{-\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta)\right\} f_X(x;\theta)\,dx \\
&= \int_{-\infty}^{\infty} \frac{2e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}\frac{e^{-(x-\theta)}}{(1+e^{-(x-\theta)})^2}\,dx \equiv \int_{-\infty}^{\infty} \frac{2e^{-2x}}{(1+e^{-x})^4}\,dx \qquad (x-\theta \longrightarrow x) \\
&= 2\int_0^1 u^2(1-u)^2\frac{1}{u(1-u)}du = \frac{1}{3} \qquad (x \longrightarrow u = e^{-x}/(1+e^{-x}))
\end{aligned}$$

5. For this pdf, the corresponding expectation is

$$\mathbb{E}_X[X;\theta] = \int_0^\infty x\frac{3\theta^3}{(x+\theta)^4}\,dx \quad = \quad \left[\frac{-x\theta^3}{(x+\theta)^3}\right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^3}\,dx = 0 + \left[-\frac{\theta^3}{2(x+\theta)^2}\right]_0^\infty = \frac{\theta}{2}$$

so therefore

$$T(\mathbf{X}) = \frac{2}{n}\sum_{i=1}^n X_i$$

is an unbiased estimator of θ. By a similar calculation

$$\begin{aligned}\mathbb{E}_X[X^2;\theta] &= \int_0^\infty x^2\frac{3\theta^3}{(x+\theta)^4}\,dx = \left[\frac{-x^2\theta^3}{(x+\theta)^3}\right]_0^\infty + \int_0^\infty \frac{2x\theta^3}{(x+\theta)^3}\,dx \\ &= 0 + \left[-\frac{x\theta^3}{(x+\theta)^2}\right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^2}\,dx = 0 + \left[-\frac{\theta^3}{(x+\theta)}\right]_0^\infty\,dx = \theta^2\end{aligned}$$

so therefore

$$\mathrm{Var}_X[X;\theta] = \theta^2 - \frac{\theta^2}{4} = \frac{3\theta^2}{4} \qquad \therefore \qquad \mathrm{Var}_T[T;\theta] = \frac{3\theta^2}{n}$$

6. In each case, we need $\mathbb{E}_{T|S}[T|S]$ (which does not depend on θ by sufficiency).

   (a) The variance of T is λ, but by symmetry

   $$T^* = \mathbb{E}_{T|S}[T|S] = \mathbb{E}[X_1|S] = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^n X_i$$

   which has variance λ/n.

   (b) We have, as T is binary,

   $$\mathbb{E}_T[T;\lambda] = \mathbb{E}_T[T^2;\lambda] = \Pr_\lambda[X_1 = 0] = e^{-\lambda} \qquad \therefore \qquad \mathrm{Var}_T[T;\lambda] = e^{-\lambda}(1 - e^{-\lambda})$$

   and

   $$T^* = \mathbb{E}_{T|S}[T|S] = \mathbb{E}[\mathbb{1}_{\{0\}}(X_1)|S] = \Pr[X_1 = 0|S]$$

   The conditional density of X_1 given S is, by independence of $X_1, \ldots, X_n$,

   $$f_{X_1|S}(x_1|s) = \frac{f_{X_1}(x_1)f_{S_{-1}}(s - x_1)}{f_S(s)}$$

   where $S_{-1} = S - X_1 \sim Poisson((n-1)\lambda)$. Therefore

   $$f_{X_1|S}(0|s) = \frac{f_{X_1}(0)f_{S_{-1}}(s)}{f_S(s)} = \frac{e^{-\lambda}(n-1)^s\lambda^s e^{-(n-1)\lambda}}{n^s\lambda^s e^{-n\lambda}} = \left(\frac{n-1}{n}\right)^s$$

   and hence

   $$T^* = \left(\frac{n-1}{n}\right)^S$$

   To compute $\mathbb{E}_{T^*}[(T^*)^k;\lambda]$, we use the Poisson probability generating function (pgf): setting $t_n = (n-1)/n$, we have

   $$\mathbb{E}_{T^*}[(T^*)^k;\lambda] = \mathbb{E}\left[\left(t_n^k\right)^S\right] = \exp\left\{n\lambda(t_n^k - 1)\right\}$$

so therefore

$$\mathbb{E}_{T^*}[T^*;\lambda] = \exp\left\{n\lambda\left(\frac{n-1}{n}-1\right)\right\} = e^{-\lambda}$$

and

$$\mathbb{E}_{T^*}[(T^*)^2;\lambda] = \exp\left\{n\lambda\left(\left(\frac{n-1}{n}\right)^2-1\right)\right\} = \exp\left\{\lambda\left(\frac{1-2n}{n}\right)\right\}$$

and therefore the variance is

$$\exp\left\{\lambda\left(\frac{1-2n}{n}\right)\right\} - e^{-2\lambda} = e^{-2\lambda}(\exp\{\lambda/n\}-1)$$

The ratio of variances is therefore

$$\frac{e^{-\lambda}(1-e^{-\lambda})}{e^{-2\lambda}(\exp\{\lambda/n\}-1)} = \frac{e^{\lambda}-1}{\exp\{\lambda/n\}-1} > 1 \qquad n > 1.$$

(c) The variance of T is $4\text{Var}_{X_1}[X_1;\theta] = 4\theta^2/12 = \theta^2/3$. Now, we need the conditional distribution $X_1|X_{(n)} = t$. Consider the theorem following decomposition with the random variable R_n recording the index of the variable in the original sample that corresponds to the maximal order statistic: we have $\Pr[R_n = 1] = 1/n$ by symmetry, so

$$\begin{aligned} f_{X_1|X_{(n)}}(x_1|t;\theta) &= f_{X_1|X_{(n)}}(x_1|t, R_n = 1;\theta)\Pr[R_n = 1] + f_{X_1|X_{(n)}}(x_1|t, R_n > 1;\theta)\Pr[R_n > 1] \\ &= \mathbb{1}_t(x_1)\frac{1}{n} + \frac{1}{t}\left(\frac{n-1}{n}\right) \end{aligned}$$

which has expectation

$$\frac{t}{n} + \frac{t}{2}\left(\frac{n-1}{n}\right) = \frac{t(n+1)}{2n}$$

Therefore

$$T^* = \mathbb{E}_{T|S}[T|S] = \mathbb{E}[2X_1|X_{(n)}] = 2\mathbb{E}[X_1|X_{(n)}] = 2\frac{X_{(n)}(n+1)}{2n} = \left(\frac{n+1}{n}\right)X_{(n)}.$$

The distribution of $X_{(n)}$ is available from results for maximum order statistics

$$f_{X_{(n)}}(x;\theta) = \frac{nx^{n-1}}{\theta^n}\mathbb{1}_{(0,\theta)}(x)$$

with, therefore

$$\mathbb{E}_{T^*}[(T^*)^k;\theta] = \frac{n}{\theta^n}\int_0^\theta x^{n+k-1}\,dx = \frac{n\theta^k}{n+k}$$

so that

$$\mathbb{E}_{T^*}[T^*;\theta] = \frac{n}{n+1}\theta \qquad \mathbb{E}_{T^*}[(T^*)^2;\theta] = \frac{n\theta^2}{n+2} \qquad \therefore \qquad \text{Var}_{T^*}[T^*;\theta] = \frac{n\theta^2}{(n+1)^2(n+2)}$$

Therefore an unbiased estimator based on T^* is $(n+1)X_{(n)}/n$, which has variance

$$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3} \qquad n > 1.$$

7. The joint pdf is

$$f_{\mathbf{X}}(\mathbf{x};\theta) = \frac{1}{\theta^n}\prod_{i=1}^n \mathbb{1}_{(\theta,2\theta)}(X_i) = \frac{1}{\theta^n}\mathbb{1}_{(0,X_{(1)})}(\theta)\mathbb{1}_{(X_{(n)}/2,\infty)}(\theta)$$

so therefore $S = (X_{(1)}, X_{(n)})$ is a jointly sufficient statistic for θ. Also, by properties of the uniform distribution

$$\mathbb{E}_T[T;\theta] = \frac{2}{3}\left(\theta + \frac{\theta}{2}\right) = \theta \qquad \text{and} \qquad \text{Var}_T[T;\theta] = \frac{4}{9}\frac{\theta^2}{12} = \frac{\theta^2}{27}.$$

By the Rao-Blackwell theorem, we can improve the variance of the estimator by conditioning on the sufficient statistic. We have

$$T^* = \mathbb{E}_{T|S}[T|S] = \mathbb{E}[2X_1/3|X_{(1)}, X_{(n)}] = \frac{2}{3}\mathbb{E}[X_1|X_{(1)}, X_{(n)}] = \frac{X_{(1)} + X_{(n)}}{3}$$

as, given $(X_{(1)}, X_{(n)})$, $X_1 \sim Uniform(X_{(1)}, X_{(n)})$. The expectation of U is computed from the distribution of the order statistics. We have that

$$\mathbb{E}_{X_{(n)}}[X_{(n)};\theta] = \theta + \frac{n}{n+1}\theta = \frac{(2n+1)\theta}{n+1}$$

and from first principles, for $\theta < x < 2\theta$, the cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x;\theta) = 1 - \left(1 - \frac{x-\theta}{\theta}\right)^n = 1 - \left(\frac{2\theta - x}{\theta}\right)^n$$

so that

$$f_{X_{(1)}}(x;\theta) = n\frac{(2\theta - x)^{n-1}}{\theta^n}\mathbb{1}_{(\theta,2\theta)}(x)$$

Therefore

$$\begin{aligned}
\mathbb{E}_{X_{(1)}}[X_{(1)};\theta] &= \frac{n}{\theta^n}\int_\theta^{2\theta} x(2\theta - x)^{n-1}\,dx = \frac{1}{\theta^n}\left[-x(2\theta - x)^n\right]_\theta^{2\theta} + \frac{1}{\theta^n}\int_\theta^{2\theta}(2\theta - x)^n\,dx \\
&= \theta + \frac{1}{\theta^n}\left[-\frac{1}{n+1}(2\theta - x)^{n+1}\right]_\theta^{2\theta} \\
&= \theta + \frac{1}{(n+1)}\theta = \frac{(n+2)\theta}{n+1}
\end{aligned}$$

Therefore T^* has expectation

$$\frac{1}{3}\left[\frac{(2n+1)\theta}{n+1} + \frac{(n+2)\theta}{n+1}\right] = \theta$$

and, by the Rao-Blackwell result, T^* has lower variance than T.

By standard methods concerning order statistics, the joint density of $(U, V) = (X_{(1)}, X(n))/\theta - 1$ is

$$f_{U,V}(u, v) = n(n-1)(v-u)^{n-2} \qquad 0 < u < v < 1$$

Setting $R = (V - U)$ and $M = (U + V)$, so that $U = (M - R)/2$ and $V = (R + M)/2$, yielding Jacobian $1/2$, and joint pdf

$$f_{M,R}(m, r) = \frac{n(n-1)}{2}r^{n-2}$$

on the triangle defined by the inequalities

$$0 < (m - r) < (m + r) < 2.$$

The marginal for m is obtained by integrating out r from the joint. For $0 < m < 2$

$$\begin{aligned} f_M(m) = \int_0^{\min\{m,2-m\}} \frac{n(n-1)}{2} r^{n-2}\,dr &= \frac{n(n-1)}{2}\left[\frac{r^{n-1}}{n-1}\right]_0^{\min\{m,2-m\}} \\ &= \frac{n}{2}(\min\{m,2-m\})^{n-1} \\ &= \begin{cases} \dfrac{n}{2}m^{n-1} & m \le 1 \\ \\ \dfrac{n}{2}(2-m)^{n-1} & 1 < m \le 2 \end{cases} \end{aligned}$$

Therefore, by symmetry $\mathbb{E}_M[M] = 1$, and

$$\begin{aligned} \mathbb{E}_M[M^2] &= \int_0^1 \frac{n}{2}m^{n+1}\,dm + \int_1^2 \frac{n}{2}m^2(2-m)^{n-1}\,dm \\ &= \frac{n}{2(n+2)} + \frac{1}{2}\left[-m^2(2-m)^n\right]_1^2 + \int_1^2 m(2-m)^n\,dm \\ &= \frac{n}{2(n+2)} + \frac{1}{2} + \left[-\frac{1}{n+1}m(2-m)^{n+1}\right]_1^2 + \frac{1}{n+1}\int_1^2 (2-m)^{n+1}\,dm \\ &= \frac{n}{2(n+2)} + \frac{1}{2} + \frac{1}{(n+1)} + \frac{1}{n+1}\left[-\frac{1}{n+2}(2-m)^{n+2}\right]_1^2 \\ &= \frac{n}{2(n+2)} + \frac{1}{2} + \frac{1}{(n+1)} + \frac{1}{n+1}\frac{1}{n+2} \\ &= \frac{n(n+1)+(n+1)(n+2)+2(n+2)+2}{2(n+1)(n+2)} = \frac{n^2+3n+4}{(n+1)(n+2)} \end{aligned}$$

Hence the variance is

$$\mathrm{Var}_M[M] = \frac{n^2+3n+4-(n+1)(n+2)}{(n+1)(n+2)} = \frac{2}{(n+1)(n+2)}$$

Now, the estimator of interest is

$$T^* = \frac{X_1+X_n}{3} = \frac{\theta(U+V)}{3} + \theta = \frac{\theta M}{3} + \theta$$

so the variance of T^* is

$$\frac{2\theta^2}{9(n+1)(n+2)} < \frac{\theta^2}{27}$$

when $n > 1$.

8. (a) By invariance, the ML estimator of $\tau(\theta)$ is

$$\widehat{\tau}(T) = \left(1 - \frac{T}{n}\right)^2 = 1 - \frac{2T}{n} + \frac{T^2}{n^2}$$

(b) We have that $T \sim Binomial(n,\theta)$, so $\mathbb{E}_T[T;\theta] = n\theta$ and $\mathbb{E}_T[T^2;\theta] = n\theta(1-\theta)+n^2\theta^2$, so

$$\mathbb{E}_T[\widehat{\tau}(T);\theta] = 1 - 2\theta + \frac{\theta(1-\theta)}{n} + \theta^2 = (1-\theta)^2 + \frac{\theta(1-\theta)}{n}$$

so the bias is

$$\frac{\theta(1-\theta)}{n}$$

(c) Again use Rao-Blackwell: the statistic $T' = \mathbb{1}_{\{0\}}(X_1+X_2)$ is unbiased for τ, as

$$\mathbb{E}_{T'}[T';\theta] = \Pr[X_1+X_2=0] = (1-\theta)^2$$

Therefore if we can compute the expectation of T' given T, then the resulting estimator T^* will have lower variance than T.

$$\mathbb{E}_{T'|T}[T'|T] = \Pr[X_1+X_2=0|T]$$

Now, for $0 \le t \le n$,

$$\begin{aligned}
\Pr[X_1+X_2=0|T=t;\theta] &= \frac{\Pr\left[X_1=0, X_2=0, \sum\limits_{i=3}^{n} X_i = t;\theta\right]}{\Pr\left[\sum\limits_{i=1}^{n} X_i = t;\theta\right]} \\
&= \frac{(1-\theta)^2\dbinom{n-2}{t}\theta^t(1-\theta)^{n-t-2}}{\dbinom{n}{t}\theta^t(1-\theta)^{n-t}} \\
&= \frac{\dbinom{n-2}{t}}{\dbinom{n}{t}} = \frac{(n-2)!}{t!(n-2-t)!}\frac{t!(n-t)!}{n!} \\
&= \frac{(n-t)(n-t-1)}{n(n-1)} = \left(1-\frac{t}{n}\right)\left(1-\frac{t}{n-1}\right)
\end{aligned}$$

Therefore the suggested estimator is

$$\left(1-\frac{T}{n}\right)\left(1-\frac{T}{n-1}\right) = 1 - \frac{T}{n} - \frac{T}{n-1} + \frac{T^2}{n(n-1)}$$

which has expectation

$$1 - \theta - \frac{n\theta}{n-1} + \frac{n\theta(1-\theta)+n^2\theta^2}{n(n-1)} = (1-\theta)^2$$

that is, the estimator

$$T^* = \left(1-\frac{T}{n}\right)\left(1-\frac{T}{n-1}\right)$$

is unbiased, so therefore must be the best unbiased estimator by Rao-Blackwell.