MATH 557 - ASSIGNMENT 4: SOLUTIONS

1. (a) Up to proportionality, the likelihood takes the form

$$\left(\frac{1}{\sigma^2}\right)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}=\left(\frac{1}{\sigma^2}\right)^{n/2}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\overline{x}_n)^2+n(\overline{x}_n-\theta)^2\right]\right\}$$

so therefore up to proportionality, the posterior, given by the product of likelihood and prior, is

$$\left(\frac{1}{\sigma^2}\right)^{(n+\alpha)/2+2}\exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^{n}(x_i-\overline{x}_n)^2+n(\overline{x}_n-\theta)^2+\lambda(\theta-\mu)^2\right]\right\}. \tag{1}$$

Completing the square in θ for the last two terms in the exponent yields

$$n(\overline{x}_n-\theta)^2+\lambda(\theta-\mu)^2=(n+\lambda)\left(\theta-\frac{n\overline{x}_n+\lambda\mu}{n+\lambda}\right)^2+\frac{n\lambda}{n+\lambda}(\overline{x}_n-\mu^2)$$

and gathering terms together, and focusing first on θ, we have that the conditional posterior given σ^2 is proportional to

$$\exp\left\{-\frac{(n+\lambda)}{2\sigma^2}\left(\theta-\frac{n\overline{x}_n+\lambda\mu}{n+\lambda}\right)^2\right\}. \tag{2}$$

Thus

$$\pi_n(\theta|\sigma^2)\equiv Normal\left(\frac{n\overline{x}_n+\lambda\mu}{n+\lambda},\frac{\sigma^2}{(n+\lambda)}\right).$$

For the marginal posterior for σ^2, we note first that

$$\pi_n(\sigma^2)=\int_{-\infty}^{\infty}\pi_n(\theta,\sigma^2)\,d\theta,$$

where the integrand is proportional to (1). Integrating wrt θ, after forming the quadratic form and using representation (2), leaves a constant of integration equal to

$$\sqrt{\frac{\sigma^2}{(n+\lambda)}}$$

yielding a marginal posterior for σ^2 proportional to

$$\left(\frac{1}{\sigma^2}\right)^{(n+\alpha)/2+1}\exp\left\{-\frac{1}{2\sigma^2}\left[\beta+\sum_{i=1}^{n}(x_i-\overline{x}_n)^2+\frac{n\lambda}{n+\lambda}(\overline{x}_n-\mu^2)\right]\right\}. \tag{3}$$

That is

$$\pi_n(\sigma^2)\equiv InvGamma\left(\frac{n+\alpha}{2},\frac{\beta+\sum\limits_{i=1}^{n}(x_i-\overline{x}_n)^2+\dfrac{n\lambda}{n+\lambda}(\overline{x}_n-\mu^2)}{2}\right).$$

Therefore, the prior structure is converted to an identical posterior structure. 6 MARKS

(b) The posterior expectation can be computed using iterated expectation over the joint distribution; given σ^2, the expectation of θ is

$$\frac{n\overline{x}_n+\lambda\mu}{n+\lambda}=w_n\overline{x}_n+(1-w_n)\mu$$

where $w_n=n/(n+\lambda)$. This expectation does not depend on σ^2. Hence the result follows as taking the expectation wrt σ^2 leaves the expectation unaltered, and as $\widehat{\theta}_n=\overline{x}_n$, and $0<w_n<1$. 4 MARKS

2. The likelihood from the model is proportional to

$$\exp\{\theta s - nK(\theta)\} \qquad s = \sum_{i=1}^{n} x_i.$$

Consider the prior $\pi_0(\theta)$ given up to proportionality by

$$\pi_0(\theta) \propto \exp\{\alpha\theta - \beta K(\theta)\}$$

normalized by dividing by the constant

$$c(\alpha, \beta) = \int_{\Theta} \exp\{\alpha\theta - \beta K(\theta)\}\, d\theta.$$

The posterior is therefore

$$\begin{aligned}\pi_n(\theta) &\propto \exp\{\theta s - nK(\theta)\} \exp\{\alpha\theta - \beta K(\theta)\} \\ &= \exp\{(\alpha + s)\theta - (\beta + n)K(\theta)\}\end{aligned}$$

which is of the same form as the prior, but with parameters updated

$$(\alpha, \beta) \longrightarrow (\alpha + s, \beta + n).$$

3. In this analysis, the ML estimate is 36/10 = 3.6

   (i) The Wald interval is given as in lectures by

$$\widehat{\theta}_n \pm \frac{1.96}{\sqrt{n}} \left\{\widehat{\theta}_n\right\}^{1/2}$$

```
s<-36
n<-10
theta.hat<-s/n
ci.wald<-theta.hat+c(-1,1)*sqrt(theta.hat)*qnorm(0.975)/sqrt(n)
ci.wald

+ [1] 2.424022 4.775978
```

   Hence the Wald interval is (2.4240216:4.7759784). 2 MARKS

   (ii) The ends of the likelihood interval are given as in lectures as solutions to the equation

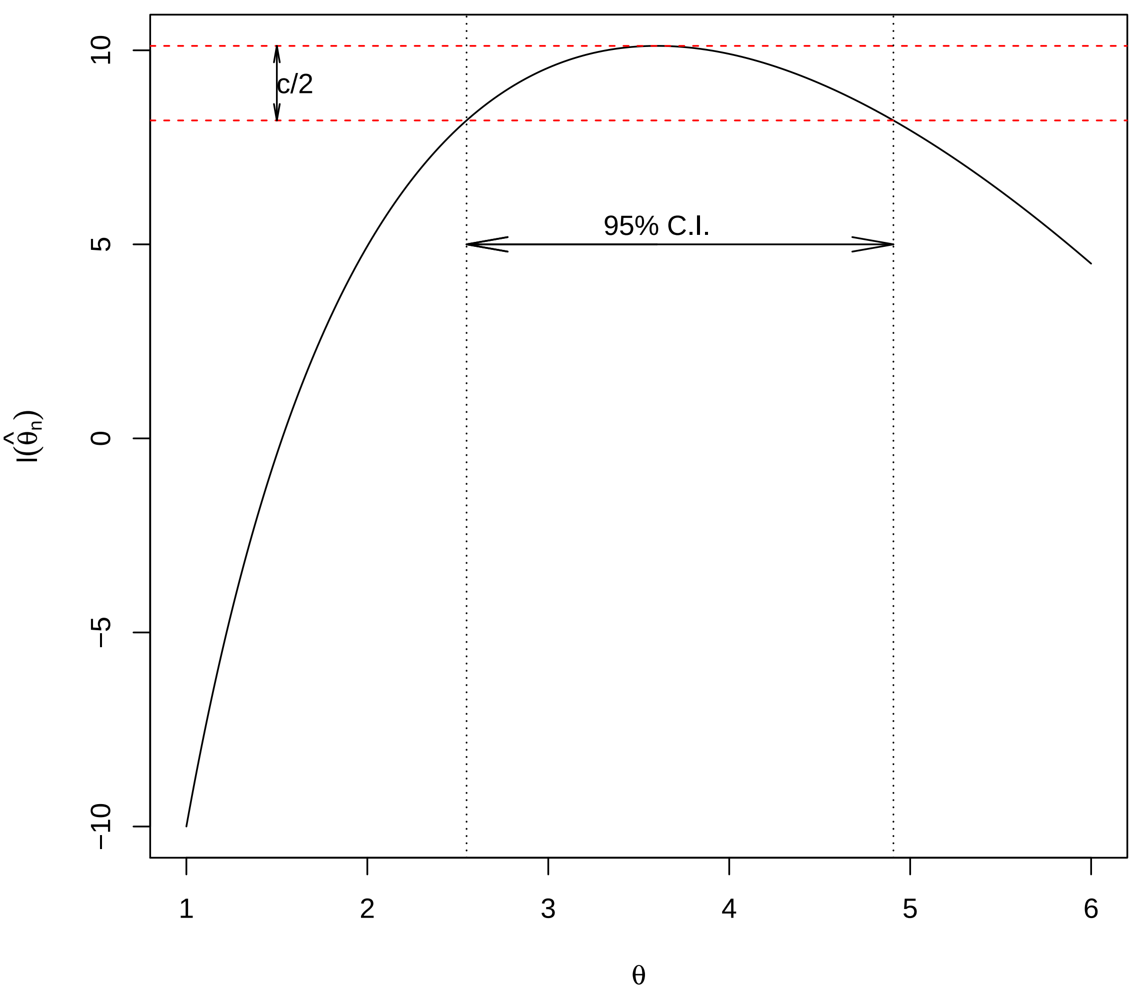
$$(s\log(\widehat{\theta}_n) - n\widehat{\theta}_n)) - (s\log(\theta) - n\theta)) = c/2$$

   where c is the 0.95 quantile of the χ^2_1 distribution, that is 3.8414588

```
(cval<-qchisq(0.95,1))

+ [1] 3.841459

pois.like<-function(x){return((s*log(theta.hat)-n*theta.hat)-(s*log(x)-n*x)-cval/2)}
ci.like.1<-uniroot(pois.like,interval=c(1,theta.hat))$root
ci.like.2<-uniroot(pois.like,interval=c(theta.hat,6))$root
c(ci.like.1,ci.like.2)

+ [1] 2.548430 4.907372
```

   Hence the likelihood interval is (2.5484296:4.9073723).

```
theta<-seq(1,6,length=1001)
log.like<-s*log(theta) - n*theta
par(mar=c(4,5,2,2))
plot(theta,log.like,type='l',xlab=expression(theta),ylab=expression(l(hat(theta)[n])))
max.like<-s*log(theta.hat) - n*theta.hat
abline(h=max.like,lty=2,col='red')
abline(h=max.like-cval/2,lty=2,col='red')
abline(v=c(ci.like.1,ci.like.2),lty=3)
arrows(theta.hat,5,ci.like.1,5,angle=10)
arrows(theta.hat,5,ci.like.2,5,angle=10)
arrows(theta.hat,5,ci.like.1,5,angle=10)
arrows(1.5,max.like,1.5,max.like-cval/2,angle=10,code=3,length=0.1)
text(1.6,max.like-cval/4,"c/2")
text(theta.hat,5.5,"95% C.I.")
```



2 Marks

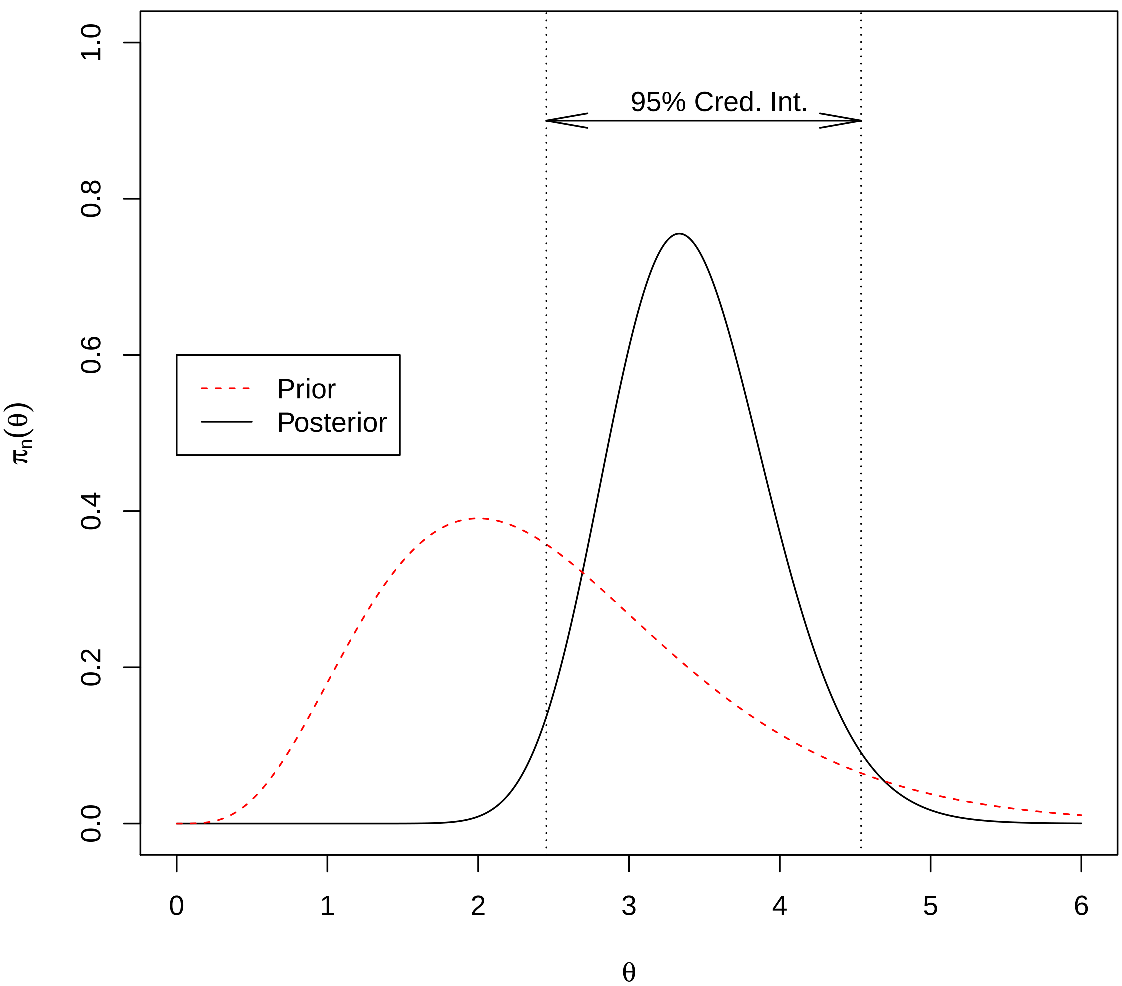
(iii) The Bayesian posterior distribution is given (as in lectures) by

$$\pi_n(\theta) \propto \theta^s \exp\{-n\theta\}\, \theta^{\alpha-1} \exp\{-\beta\theta\} = \theta^{(\alpha+s)-1} \exp\{-(\beta+n)\theta\}$$

that is, the posterior is $Gamma((\alpha+s),(\beta+n)) \equiv Gamma(5+36, 2+10) \equiv Gamma(41, 12)$.

```
theta<-seq(0,6,length=1001)
prior.dens<-dgamma(theta,5,2)
posterior.dens<-dgamma(theta,41,12)
```

```
par(mar=c(4,5,2,2))
plot(theta,posterior.dens,type='l',xlab=expression(theta),
     ylab=expression(pi[n](theta)),ylim=range(0,1))
lines(theta,prior.dens,lty=2,col='red')
ci.bayes.1<-qgamma(0.025,41,12)
ci.bayes.2<-qgamma(0.975,41,12)
c(ci.bayes.1,ci.bayes.2)

+ [1] 2.451859 4.539054

abline(v=c(ci.bayes.1,ci.bayes.2),lty=3)
legend(0,0.6,c('Prior','Posterior'),lty=c(2,1),col=c('red','black'))
arrows(ci.bayes.1,0.9,ci.bayes.2,0.9,angle=10,code=3)
text(theta.hat,0.925,"95% Cred. Int.")
```



Hence the Bayesian credible interval is (2.451859:4.5390539).

2 MARKS