MATH 557 - ASSIGNMENT 3
SOLUTIONS

(a) By standard arguments, we have that the maximum likelihood estimator (MLE) of θ_0 is $\widehat{\theta}_n = \overline{X}_n$, the sample mean: the log-likelihood is

$$\ell_n(\theta) = \text{const.} - \frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2 = \text{const.} - \frac{n}{2}(\overline{x}_n - \theta)^2$$

and the quadratic term is minimized wrt θ at $\overline{x}_n$. Therefore, by invariance, the MLE of

$$\tau_c(\theta_0) = \Pr_{\theta_0}[X \leq c] = \Phi(c - \theta_0)$$

is

$$\widehat{\tau}_c(\theta_0) = \Phi(c - \widehat{\theta}_n) = \Phi(c - \overline{X}_n).$$

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(b) For the Taylor expansion to 4th order of $\Phi(c - t)$, around $t = \theta_0$, we have

$$\Phi(c-t) = \Phi(c-\theta_0) + (t-\theta_0)\dot{\Phi}(c-\theta_0) + \frac{1}{2}(t-\theta_0)^2\ddot{\Phi}(c-\theta_0) + \frac{1}{6}(t-\theta_0)^3\dddot{\Phi}(c-\theta_0) + \frac{1}{24}(t-\theta_0)^4 R(t,\theta_0)$$

where the remainder term $R(t, \theta_0)$ depends on the fourth derivative of Φ evaluated at a point in $\mathbb{R}$ between t and θ_0. We have by direct calculation that

$$\begin{aligned}
\dot{\Phi}(x) &= \phi(x) \\
\ddot{\Phi}(x) &= \dot{\phi}(x) = -x\phi(x) \\
\dddot{\Phi}(x) &= \ddot{\phi}(x) = (x^2 - 1)\phi(x) \\
\ddddot{\Phi}(x) &= \dddot{\phi}(x) = -x(x^2 + 3)\phi(x)
\end{aligned}$$

therefore, setting $t = \overline{X}_n$, we have

$$\begin{aligned}
\Phi(c - \overline{X}_n) = \Phi(c - \theta_0) - (\overline{X}_n - \theta_0)\phi(c - \theta_0) + \frac{1}{2}(\overline{X}_n - \theta_0)^2\dot{\phi}(c - \theta_0) \\
- \frac{1}{6}(\overline{X}_n - \theta_0)^3\ddot{\phi}(c - \theta_0) + \frac{1}{24}(\overline{X}_n - \theta_0)^4\dddot{\phi}(L_n)
\end{aligned}$$

where L_n is a random quantity that lies between $c - \theta_0$ and $c - \overline{X}_n$. Taking expectations through this expression yields that

$$\mathbb{E}_{X_{1:n}}[\Phi(c - \overline{X}_n); \theta_0] = \Phi(c - \theta_0) + \frac{1}{2n}\dot{\phi}(c - \theta_0) + \frac{1}{24}\mathbb{E}_{X_{1:n}}[(\overline{X}_n - \theta_0)^4\dddot{\phi}(Z); \theta_0]$$

as $\overline{X}_n \sim Normal(\theta_0, 1/n)$, so

$$\mathbb{E}_{X_{1:n}}[(\overline{X}_n - \theta_0)^r; \theta_0] = 0, \qquad r = 1, 3, 5, \ldots$$

and

$$\mathbb{E}_{X_{1:n}}[(\overline{X}_n - \theta_0)^2; \theta_0] \equiv \text{Var}_{X_{1:n}}[\overline{X}_n; \theta_0] = \frac{1}{n}.$$

For the fourth order term, it is clear that $\ddddot{\Phi}(x) = \dddot{\phi}(x)$ is a bounded function, as the exponential of the quadratic in x dominates when x is large and negative or large and positive; say $\dddot{\phi}(x) < M$ for some finite M. Therefore

$$\mathbb{E}_{X_{1:n}}[(\overline{X}_n - \theta_0)^4 \dddot{\phi}(L_n); \theta_0] < M\mathbb{E}_{X_{1:n}}[(\overline{X}_n - \theta_0)^4; \theta_0] = \frac{3M}{n^2}$$

as $Z = \sqrt{n}(\overline{X}_n - \theta_0) \sim Normal(0, 1)$, and $\mathbb{E}_Z[Z^4] = 3$. Therefore the expectation of $\widehat{\tau}_c(\theta_0) = \Phi(c - \widehat{\theta}_n)$ is bounded by

$$\Phi(c - \theta_0) + \frac{1}{2n}\dot{\phi}(c - \theta_0) + \frac{M}{8n^2} \tag{1}$$

and noting the earlier result for $\dot{\phi}(x)$, the bias is therefore

$$-\frac{1}{2n}\dot{\phi}(c - \theta_0) + \mathrm{O}(n^{-2}) = -\frac{1}{2n}(c - \theta_0)\phi(c - \theta_0) + \mathrm{O}(n^{-2}).$$

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(c) For the variance, we need to study the second moment: we have

$$\{\Phi(c - t)\}^2 = \{\Phi(c - \theta_0)\}^2 - 2(t - \theta_0)\phi(c - \theta_0)\Phi(c - \theta_0) + (t - \theta_0)^2 \left[\{\phi(c - \theta_0)\}^2 + \dot{\phi}(c - \theta_0)\Phi(c - \theta_0)\right] + \ldots$$

and evaluating at $t = \overline{X}_n$ and taking expectations wrt the distribution of $\overline{X}_n$ yields

$$\mathbb{E}_{X_{1:n}}[\{\Phi(c - \overline{X}_n)\}^2; \theta_0] = \{\Phi(c - \theta_0)\}^2 + \frac{1}{n}\left[\{\phi(c - \theta_0)\}^2 + \dot{\phi}(c - \theta_0)\Phi(c - \theta_0)\right] + \mathrm{O}(n^{-2}). \tag{2}$$

Hence, the variance is obtained by taking the difference between (2) and the square of (1):

$$\{\Phi(c - \theta_0)\}^2 + \frac{1}{n}\left[\{\phi(c - \theta_0)\}^2 + \dot{\phi}(c - \theta_0)\Phi(c - \theta_0)\right] - \left(\Phi(c - \theta_0) + \frac{1}{2n}\dot{\phi}(c - \theta_0) + \frac{M}{8n^2}\right)^2$$

which simplifies to

$$\frac{1}{n}\{\phi(c - \theta_0)\}^2 + \mathrm{O}(n^{-2})$$

after cancellation.

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Important Note: we have demonstrated that the bias and variance of $\widehat{\tau}_n$ are both $\mathrm{O}(n^{-1})$, so therefore the mean-square error (MSE)

$$\text{MSE} = (\text{Bias})^2 + \text{Variance}$$

has order $\mathrm{O}(n^{-1})$, and is dominated by the Variance term.

(d) Here we may use the Delta Method: the mapping concerned is

$$g(t) = \Phi(c - t) \qquad \dot{g}(t) = \phi(c - t)$$

so therefore

$$\sqrt{n}(\widehat{\tau}_n - \tau_c(\theta_0)) \xrightarrow{d} Normal(0, \{\phi(c - \theta_0)\}^2).$$

Note also that

$$\lim_{n \longrightarrow \infty} n\text{Var}_{X_{1:n}}[\widehat{\tau}_n]$$

is equal to the asymptotic variance as implied by the Delta Method.

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