557: Mathematical Statistics II
Complete Statistics in the Exponential Family

Suppose that f is a one-parameter natural Exponential Family distribution in canonical form, written using the "tilting" formulation as

$$f(x;\theta) = f_0(x)\exp\{\theta x - K(\theta)\}$$

for pdf $f_0(x)$, for all θ in some open interval. $K(\theta)$ is the *cumulant generating function*:

$$\int_{-\infty}^{\infty} f_0(x)\exp\{\theta x\}\,dx = e^{K(\theta)}.$$

EXAMPLE: $Normal(\theta, 1)$

$$f(x;\theta) = \left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{-\frac{1}{2}(x-\theta)^2\right\} = f_0(x)\exp\{\theta x - K(\theta)\}$$

where

$$f_0(x) = \left(\frac{1}{2\pi}\right)^{1/2}\exp\left\{-\frac{1}{2}x^2\right\} \qquad\qquad K(\theta) = \frac{t^2}{2}$$

Suppose that there exists $g(x)$ such that

$$\int_{-\infty}^{\infty} g(x)f(x;\theta)\,dx = \int_{-\infty}^{\infty} g(x)f_0(x)\exp\{\theta x - K(\theta)\}\,dx = 0 \tag{1}$$

for all θ. Write

$$g(x) = g_+(x) - g_-(x)$$

where

$$g_+(x) = \max\{0, g(x)\} \qquad\qquad g_-(x) = \max\{0, -g(x)\}$$

are the positive and negative part functions. Note that $g_+(x) \geq 0$ and $g_-(x) \geq 0$ for all x. Thus for a specific value θ_0, multiplying equation (1) by $e^{K(\theta)}$ and rearranging, we have

$$\int_{-\infty}^{\infty} g_+(x)f_0(x)\exp\{\theta x\}\,dx = \int_{-\infty}^{\infty} g_-(x)f_0(x)\exp\{\theta x\}\,dx = c(\theta) \geq 0 \tag{2}$$

for all θ in a neighbourhood of θ_0. At θ_0, write $c(\theta_0) = c_0$. If $c_0 = 0$, then we must have

$$g_+(x) = g_-(x) = g(x) = 0$$

for all x, as all terms in the integrands are non-negative. If $c_0 > 0$, the functions

$$f_+(x) = \frac{g_+(x)f_0(x)}{c_0}\exp\{\theta_0 x\} \qquad\qquad f_-(x) = \frac{g_-(x)f_0(x)}{c_0}\exp\{\theta_0 x\}$$

are probability densities in x, and the moment generating function of f_+ is

$$M_+(t) = \int_{-\infty}^{\infty} e^{tx}f_+(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\frac{g_+(x)f_0(x)}{c_0}\exp\{\theta_0 x\}\,dx = \int_{-\infty}^{\infty} \frac{g_+(x)f_0(x)}{c_0}\exp\{(\theta_0+t)x\}\,dx$$

and similarly

$$M_-(t) = \int_{-\infty}^{\infty} \frac{g_-(x)f_0(x)}{c_0}\exp\{(\theta_0+t)x\}\,dx.$$

But $\theta_0 + t$ is in a neighbourhood of θ_0 for t in a neighbourhood of zero, and thus by equation (2), it follows that

$$M_+(t) = M_-(t) = \frac{c(\theta_0+t)}{c(\theta_0)}.$$

Due to the uniqueness of mgfs, this implies that $f_+(x) = f_-(x)$ for all x, which consequently implies that $g_+(x) = g_-(x)$ for all x. Therefore we must have that $g(x) = g_+(x) - g_-(x) \equiv 0$ for all x. Hence X is complete.