557: Mathematical Statistics II
Sufficiency, Ancillarity and Completeness

For a random sample $\mathbf{X}$ from a probability model with pmf/pdf $f_{\mathbf{X}}(\mathbf{x};\theta)$. Let $\mathbf{x} \in \mathbb{X}^n \subseteq \mathbb{R}^n$ be a possible realization of $\mathbf{X}$.

- **Sufficiency**

  A statistic, $\mathbf{T}(\mathbf{X})$, is **sufficient** for θ (or θ-sufficient) if the conditional distribution of $\mathbf{X}$ given $\mathbf{T}(\mathbf{X})$ does not depend on θ.

- **Minimal Sufficiency**

  A statistic, $\mathbf{T}(\mathbf{X})$, is **minimal sufficient** for θ if it is a sufficient statistic, and for any other sufficient statistic $\mathbf{T}^*(\mathbf{X})$, $\mathbf{T}$ is a function of $\mathbf{T}^*$, that is, for $\mathbf{x}, \mathbf{y} \in \mathbb{X}^n$,

  $$\mathbf{T}^*(\mathbf{x}) = \mathbf{T}^*(\mathbf{y}) \quad \Longrightarrow \quad \mathbf{T}(\mathbf{x}) = \mathbf{T}(\mathbf{y})$$

  A sufficient condition for minimal sufficiency is that for two realizations $\mathbf{x} = x_{1:n}$ and $\mathbf{y} = y_{1:n}$,

  $$T(\mathbf{x}) = T(\mathbf{y}) \qquad \Longleftrightarrow \qquad \frac{f_{\mathbf{X}}(\mathbf{x};\theta)}{f_{\mathbf{X}}(\mathbf{y};\theta)} \text{ does not depend on } \theta.$$

- **Ancillarity**

  A statistic, $\mathbf{S}(\mathbf{X})$, is **ancillary** for θ if the conditional distribution of $\mathbf{S}(\mathbf{X})$ does not depend on θ.

- **Completeness**

  A statistic, $\mathbf{T}$, with pmf/pdf $f_{\mathbf{T}}(\mathbf{t};\theta)$ is **complete** if, for every real-valued (technically, measurable with respect to the corresponding probability measure, P_θ) function $\mathbf{g}$,

  $$\mathbb{E}_{\mathbf{T}}[\mathbf{g}(\mathbf{T});\theta] = \mathbf{0} \quad \Longrightarrow \quad P_\theta[\, \mathbf{g}(\mathbf{T}) = \mathbf{0}\,] = 1$$

  that is $\mathbf{g}(\mathbf{T}) \equiv 0$ with probability 1 for all $\theta \in \Theta$.

  $\mathbf{T}$ is **boundedly complete** if, for every **bounded** real-valued function $\mathbf{g}$,

  $$\mathbb{E}_{\mathbf{T}}[\mathbf{g}(\mathbf{T});\theta] = \mathbf{0} \quad \Longrightarrow \quad P[\, \mathbf{g}(\mathbf{T}) = \mathbf{0};\theta] = 1.$$