MATH 557 - MID-TERM EXAMINATION

Marks can be obtained by answering all questions.
40 marks available. Rescaling of the final mark may occur.

1. Derive (giving details) sufficient statistics for random samples of size n from the following distributions. Note that the sufficient statistics may be multidimensional, but must have dimension no greater than two.

   (a) The Beta density with parameters α and β

   $$f_X(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

   and zero otherwise, for $\alpha, \beta > 0$.

   3 MARKS

   (b) The pmf

   $$f_X(x; \theta) = \frac{(\log\theta)^x}{\theta x!} \qquad x = 0, 1, 2, \ldots$$

   and zero otherwise, for $\theta > 1$.

   3 MARKS

   (c) The pdf

   $$f_X(x; \theta) = \frac{1}{\theta}\exp\{-(x-\theta)/\theta\} \qquad x > \theta$$

   and zero otherwise, for some $\theta > 0$.

   4 MARKS

2. Consider the model with pdf

   $$f_X(x; \theta, \sigma) = \exp\left\{-\left(\frac{x-\theta}{\sigma}\right)^4 - \kappa(\theta, \sigma)\right\} \qquad -\infty < x < \infty$$

   for $\theta \in \mathbb{R}$ and $\sigma > 0$, for some function $\kappa(.,.)$.

   (a) Find a minimal sufficient statistic for parameters $(\theta, \sigma)^\top$ based on a random sample of size n, $X_1, \ldots, X_n$.

   6 MARKS

   (b) Suppose that $\sigma = 1$ is known. Find a (scalar) statistic that is ancillary with respect to θ.

   4 MARKS

3. Find the maximum likelihood estimates of parameters (from a random sample of size n) in the following continuous probability models for $x \in \mathbb{R}$. The function $\mathbb{1}_A(x)$ is the indicator function for the set A.

   (a) The model with pdf

   $$f_X(x;\theta) = \mathbb{1}_{(0,1)}(x)\theta(1-x)^{\theta-1}$$

   for $\theta > 0$.

   5 MARKS

   (b) The model with pdf

   $$f_X(x;\alpha,\beta) = \mathbb{1}_{(0,\beta)}(x)\frac{\alpha x^{\alpha-1}}{\beta^\alpha}$$

   for $\alpha, \beta > 0$. 5 MARKS

4. This question focusses on the pdf defined for $x \in \mathbb{R}$ by

$$f_X(x;\theta_1,\theta_2) = \frac{1}{\theta_1+\theta_2}\left(\mathbb{1}_{(-\infty,0]}(x)\exp\left\{\frac{x}{\theta_2}\right\} + \mathbb{1}_{(0,\infty)}(x)\exp\left\{-\frac{x}{\theta_1}\right\}\right)$$

where $\theta_1, \theta_2 > 0$. A random sample of size n from this pdf is available.

   (a) Show that the maximum likelihood estimator of $\theta = (\theta_1, \theta_2)^\top$ is a function of the statistics T_1 and T_2, where

   $$T_1 = \sum_{i=1}^n \mathbb{1}_{(0,\infty)}(X_i)X_i \qquad T_2 = -\sum_{i=1}^n \mathbb{1}_{(-\infty,0]}(X_i)X_i$$

   6 MARKS

   (b) Assuming correct specification, find the Fisher Information quantity, $\mathcal{I}_{\theta_0}(\theta_0)$, defined by

   $$\mathcal{I}_{\theta_0}(\theta_0) = \mathbb{E}_X\left[-\frac{\partial^2}{\partial\theta\partial\theta^\top}\{\log f_X(X;\theta)\}_{\theta=\theta_0};\theta_0\right]$$

   where $\theta_0 = (\theta_{01}, \theta_{02})^\top$ is the true value of θ.

   4 MARKS