MATH 557 - PRACTICE MID-TERM EXAMINATION

Marks can be obtained by answering all questions.
40 marks available. Rescaling of the final mark may occur.

1. (a) Suppose that $X \sim Uniform(-\theta, \theta)$ for some $\theta > 0$. Let $Y = |X|$. Is Y sufficient for θ ? Justify your answer. *4 Marks*

   (b) Suppose that $X_1, \ldots, X_n$ are a random sample from the $Uniform(-\theta, 2\theta)$ distribution. Find the maximum likelihood estimator of θ, $\widehat{\theta} \equiv \widehat{\theta}_{ML}(\mathbf{X})$. Is $\widehat{\theta}_{ML}(\mathbf{X})$ sufficient for θ ? Justify your answer.

   6 Marks

2. (a) Show that the family $Normal(0, \theta)$ for $0 < \theta < \infty$ is ***not complete***.

   4 Marks

   (b) Consider a random sample $X_1, \ldots, X_n$ from the pdf

   $$f_X(x; \theta_1, \theta_2) = \frac{1}{\theta_2} \exp\{-(x - \theta_1)/\theta_2\} \mathbb{1}_{(\theta_1, \infty)}(x)$$

   for $-\infty < \theta_1 < \infty$, $\theta_2 > 0$. Find the (two-dimensional) minimal sufficient statistic for the vector (θ_1, θ_2). *6 Marks*

3. Suppose that $X_1, \ldots, X_n$ is a random sample from a $Poisson(\lambda)$ distribution.

   (a) Find the maximum likelihood estimator of $\tau(\lambda) = \{\mathrm{P}_\lambda[X_1 = 0]\}^2$

   4 Marks

   (b) Consider the estimator of $\tau(\lambda)$, $\widehat{\tau}_1(\mathbf{X}) = (-1)^{X_1}$. Show that $\widehat{\tau}_1(\mathbf{X})$ is *unbiased* for τ, that is

   $$\mathbb{E}_{\mathbf{X}}[\widehat{\tau}_1(\mathbf{X}); \lambda] = \tau(\lambda).$$

   4 Marks

   (c) Comment on $\widehat{\tau}_1(\mathbf{X})$ as an estimator of τ.

   2 Marks

4. (a) Suppose that $X_1, \ldots, X_n$ is a random sample from a $Normal(\theta, 1)$ distribution. Consider parameter $\phi(\theta) = \mathrm{P}_\theta[X \leq c]$ for some fixed c. Find an *unbiased* estimator for $\phi(\theta)$, that is, and estimator $T = T(\mathbf{X})$ for which $\mathbb{E}_T[T; \theta] = \phi(\theta)$.

   5 Marks

   (b) Suppose that $X_1, \ldots, X_n$ is a random sample from a continuous distribution with cdf $F_X(x)$. Let $\theta = \mathrm{P}[X \leq a]$ for some fixed a. Show that the estimator

   $$T(\mathbf{X}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{(-\infty, a]}(X_i)$$

   is unbiased for θ, and find $\mathrm{Var}_T[T; \theta]$.

   Hint: Consider the random variables $Y_i = \mathbb{1}_{(-\infty, a]}(X_i)$, $i = 1, \ldots, n$ and their distribution.

   5 Marks