MATH 557 - EXERCISES 4

These exercises are not for assessment

1. Let $X_1, \ldots, X_n$ be a random sample from the $Beta(1, \theta)$ probability model, for parameter $\theta > 0$.

   (a) Find the Uniformly Most Powerful (UMP) level α test (that is, the form of the test statistic and rejection region) of hypotheses

   $$\begin{aligned} H_0 &: \theta = 1 \\ H_1 &: \theta > 1 \end{aligned}$$

   (b) Find the Likelihood Ratio Test (LRT) for testing

   $$\begin{aligned} H_0 &: \theta = 1 \\ H_1 &: \theta \neq 1 \end{aligned}$$

   that has level α.

2. Find the UMP level α test (that is, the form of the test statistic and rejection region) for hypotheses

   $$\begin{aligned} H_0 &: \theta \leq \theta_0 \\ H_1 &: \theta > \theta_0 \end{aligned}$$

   where $\theta > 0$, and θ_0 is a fixed positive constant, based on a random sample of size n from the following probability models:

   (a) Exponential$(1/\theta)$:

   $$f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x > 0$$

   (b) Normal$(1, \theta)$:

   $$f_X(x;\theta) = \left(\frac{1}{2\pi\theta}\right)^{1/2} \exp\left\{-\frac{(x-1)^2}{2\theta}\right\} \qquad -\infty < x < \infty$$

3. Suppose that $X_1, \ldots, X_n \sim$ Poisson(θ) for $\theta > 0$ is a random sample. Construct a test of the hypotheses

   $$\begin{aligned} H_0 &: \theta \leq 2 \\ H_1 &: \theta > 2 \end{aligned}$$

   that is a UMP level $\alpha = 0.05$ test, that is, where

   $$\alpha = \Pr[T(\mathbf{X}) \in \mathcal{R}_T; \theta]$$

   for suitably chosen test statistic $T(\mathbf{X})$ and rejection region $\mathcal{R}_T$. Report the outcome of the test for the data set

   $$2 \quad 3 \quad 5 \quad 1 \quad 5 \quad 2.$$

   Note: As the X values are discrete, the test required here is a *randomized* test. $T(\mathbf{X})$ is also discrete, and as the critical region takes the form $T(\mathbf{x}) \geq c$, where c is an integer, to match the required level α, the test function must take the form

   $$\phi_{\mathcal{R}_T}(t) = \begin{cases} 1 & t > c \\ \gamma & t = c \\ 0 & t < c \end{cases}$$

   with γ chosen so that $\mathbb{E}_T[\phi_{\mathcal{R}_T}(T)] = \alpha$. That is, if $T(\mathbf{x}) = c$, we reject with probability γ.

4. Suppose that $X_1, \ldots, X_n \sim$ Bernoulli(θ) are a random sample, where $0 < \theta < 1$, and suppose that $\tau(\theta) = \theta(1-\theta)$.

   (a) Find the maximum likelihood estimator of $\tau(\theta)$, $\widehat{\tau}_n(\mathbf{X})$.

   (b) Find large sample approximation to the distribution of $\widehat{\tau}_n(\mathbf{X})$ for each $\theta \in (0,1)$.

5. Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution with pdf f_X, with

$$\mathbb{E}_X[X_i; \mu, \gamma] = \mu \qquad \mathrm{Var}_X[X_i; \mu, \gamma] = 1 \qquad \mathrm{Var}_X[X_i^2; \mu, \gamma] = \gamma \qquad \mathbb{E}_X[X_i^4; \mu, \gamma] < \infty$$

   for $i = 1, \ldots, n$. Denote by

$$T_{1n}(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 1 \qquad T_{2n}(\mathbf{X}) = \overline{X}^2 - \frac{1}{n}$$

   two estimators of $\tau(\mu) = \mu^2$.

   (a) Compute the *asymptotic bias* of each of these estimators.

   (b) The *Asymptotic Relative Efficiency* (ARE) of T_{1n} with respect to T_{2n} is defined as the ratio of their rescaled asymptotic mean-square errors (AMSE)

$$\mathrm{ARE}_\mu(T_{1n}, T_{2n}) = \frac{\mathrm{AMSE}_\mu(T_{2n})}{\mathrm{AMSE}_\mu(T_{1n})}$$

   where

$$\mathrm{AMSE}_\mu(T_{jn}) = \lim_{n \longrightarrow \infty} n\mathbb{E}_{f_{T_{jn}|\mu}}[(T_{jn} - \tau(\mu))^2] \qquad j = 1, 2$$

   Find $\mathrm{ARE}_\mu(T_{1n}, T_{2n})$.

6. Suppose that $(X_i, Y_i), i = 1, \ldots, n$ are independent pairs of random variables with joint pdf

$$f_{X,Y}(x, y; \theta, \phi) = \phi^2\theta \exp\{-[\phi x + \theta\phi y]\} \qquad x, y > 0$$

   for parameters $\theta, \phi > 0$. Find the maximum likelihood estimator, $\widehat{\boldsymbol{\theta}}_n$, of $\boldsymbol{\theta} = (\theta, \phi)^\top$, and a large sample approximation its distribution, given in this regular case by

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0) \stackrel{d}{\longrightarrow} Z \sim Normal(\mathbf{0}_2, \{\mathcal{I}_{\boldsymbol{\theta}_0}(\boldsymbol{\theta}_0)\}^{-1})$$

   where $\mathcal{I}_{\boldsymbol{\theta}_0}(\boldsymbol{\theta}_0)$ is the Fisher Information.

7. Derive the forms of the Wald, Score and Likelihood Ratio statistics for testing

$$\begin{aligned} H_0 &: \lambda = \lambda_0 \\ H_1 &: \lambda \neq \lambda_0 \end{aligned}$$

   if the data follow a Poisson distribution with parameter $\lambda > 0$. For each case, use the correspondence with test statistic acceptance regions to construct approximate $1 - \alpha$ confidence intervals.