MATH 557 - EXERCISES 3

These exercises are not for assessment

1 Suppose that $X \sim Binomial(n, \theta)$ for $0 < \theta < 1$.

(a) Verify that the estimator $T(X) = X/n$ is unbiased for θ.

(b) Consider $\tau(\theta) = 1/\theta$. Does there exist an unbiased estimator of $\tau(\theta)$? Justify your answer.

2 Suppose that $X_1, \ldots, X_n \sim Uniform(0, \theta)$ for $\theta > 0$ is a random sample.

(a) Find an unbiased estimator, $T(\mathbf{X})$, of θ.

(b) Compute the variance/mean square error of $T(\mathbf{X})$.

(c) It can be shown that the variance of an unbiased estimator of parameter θ derived from a random sample of size n can be no smaller than the bound

$$\frac{1}{n}\{\mathcal{I}_\theta(\theta)\}^{-1}$$

where $\mathcal{I}_\theta(\theta)$ is the Fisher information; this is termed the *Cramér-Rao lower bound*. Any unbiased estimator that achieves the variance bound is termed the *best* or *minimum variance* unbiased estimator.

Compute the amount by which the variance of $T(\mathbf{X})$ exceeds the Cramér-Rao lower bound for this non-regular model.

3 Suppose that $X_1, \ldots, X_n \sim Beta(\theta, 1)$ for $\theta > 0$ is a random sample, so that

$$f_X(x;\theta) = \theta x^{\theta-1} \qquad 0 < x < 1$$

and zero otherwise.

(a) Find the maximum likelihood estimator of θ, to be denoted $\widehat{\theta}_n(\mathbf{X})$.

(b) Is $\widehat{\theta}_n(\mathbf{X})$ unbiased for θ ? Justify your answer.

(c) Does $\widehat{\theta}_n(\mathbf{X})$ attain the Cramér-Rao lower bound ? Justify your answer.

4 Suppose that $f_X(x;\theta) = f(x-\theta)$ is a location family distribution.

(a) Show that for this class of models, the Fisher information for θ, $\mathcal{I}_\theta(\theta)$, does not depend on θ.

(b) Compute $\mathcal{I}_\theta(\theta)$ if

$$f(x) = \frac{1}{2}e^{-|x|} \qquad -\infty < x < \infty$$

(c) Compute $\mathcal{I}_\theta(\theta)$ if

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2} \qquad -\infty < x < \infty$$

5 Suppose that $X_1, \ldots, X_n$ is a random sample from the distribution with pdf

$$f_X(x;\theta) = \frac{3\theta^3}{(x+\theta)^4} \qquad 0 < x < \infty$$

and zero otherwise, for parameter $\theta > 0$.

Find an unbiased estimator for θ, and the variance of this estimator.

THE RAO-BLACKWELL THEOREM

Theorem. ***The Rao-Blackwell Theorem***

Let T be an unbiased estimator of $\tau(\theta)$, and S be a sufficient statistic for θ. Define statistic T^* by

$$T^* \equiv g(S) = \mathbb{E}_{T|S}[T|S;\theta]$$

Then T^* is an unbiased estimator of $\tau(\theta)$, and for all θ

$$Var_{T^*}[T^*;\theta] \leq Var_T[T;\theta].$$

6 Using the Rao-Blackwell Theorem in the following examples, demonstrate the construction of an estimator T^* defined by $T^* = \mathbb{E}[T|S] = g(S)$, where T is an unbiased estimator of $\tau(\theta)$, and S is sufficient for θ, which is unbiased and has variance not greater than T.

(a) If $X_1, \ldots, X_n \sim Poisson(\lambda)$, $\tau(\lambda) = \lambda$

$$S = \sum_{i=1}^{n} X_i \qquad\qquad T = X_1$$

(b) If $X_1, \ldots, X_n \sim Poisson(\lambda)$, $\tau(\lambda) = e^{-\lambda}$

$$S = \sum_{i=1}^{n} X_i \qquad\qquad T = \mathbb{1}_{\{0\}}(X_1)$$

(c) If $X_1, \ldots, X_n \sim Uniform(0,\theta)$, $\tau(\theta) = \theta$

$$S = \max\{X_1, \ldots, X_n\} \qquad\qquad T = 2X_1$$

Demonstrate the improvement in variance using simulation if necessary.

7 Suppose $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $(\theta, 2\theta)$. Find a (minimal) sufficient statistic S, and that $T = 2X_1/3$ is an unbiased estimator of θ. Using the Rao-Blackwell Theorem, find an unbiased estimator of θ with variance no greater than T.

8 Suppose $X_1, \ldots, X_n$ are independent random variables identically distributed as $Bernoulli(\theta)$. Let $\tau(\theta) = (1-\theta)^2$.

(a) Find the ML estimator of $\tau(\theta)$ as a function of the τ-sufficient statistic $T = \sum_{i=1}^{n} X_i$.

(b) Show that T is biased for τ.

(c) Find the *best* (or *minimum variance*) unbiased estimator of τ.