MATH 557 - EXERCISES 2

These exercises are not for assessment

1. Suppose that $X_1 \sim Binomial(n_1, \theta_1)$ and $X_2 \sim Binomial(n_2, \theta_2)$ be independent random variables. Derive the maximum likelihood estimator of the odds ratio ψ defined by

$$\psi = \frac{\theta_1/(1-\theta_1)}{\theta_2/(1-\theta_2)}.$$

2. Suppose that $X_1, ..., X_n$ are a random sample from a $Gamma(\alpha, \beta)$ distribution. Find the *method of moments estimators* of α and β, that is, the estimators $\widehat{\alpha}$ and $\widehat{\beta}$ defined by equating the first two moments of the distribution, $\mathbb{E}_X[X]$ and $\mathbb{E}_X[X^2]$ to the first two empirical moments

$$\frac{1}{n}\sum_{i=1}^{n} x_i \qquad\qquad \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

respectively.

3. Find the maximum likelihood estimators of the unknown parameters in the following probability densities on the basis of a random sample of size n.

   (i) $f_X(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1, \theta > 0.$

   (ii) $f_X(x;\theta) = (\theta+1)x^{-\theta-2},\ 1 < x, \theta > 0.$

   (iii) $f_X(x;\theta) = \theta^2 x \exp\{-\theta x\},\ 0 < x, \theta > 0.$

   (iv) $f_X(x;\theta) = 2\theta^2 x^{-3},\ \theta \le x, \theta > 0.$

   (v) $f_X(x;\theta) = \dfrac{\theta}{2}\exp\{-\theta\,|x|\},\ -\infty < x < \infty,\ \ \theta > 0.$

   (vi) $f_X(x;\theta_1,\theta_2) = \dfrac{1}{\theta_2 - \theta_1},\ \theta_1 \le x \le \theta_2.$

   (vii) $f_X(x;\theta_1,\theta_2) = \theta_1\theta_2^{\theta_1}x^{-\theta_1-1},\ \theta_2 \le x,\ \ \theta_1, \theta_2 > 0.$

4. An estimator, T, is an *unbiased* estimator of function $\tau(\theta)$ of parameter θ if

$$\mathbb{E}_T[T;\theta] = \tau(\theta)$$

where f_T is the *sampling distribution* of T. The *bias*, $b_\theta(T)$, and *Mean Squared Error*, $\mathrm{MSE}_\theta(T)$, of an estimator T of $\tau(\theta)$ are defined respectively by

$$b_\theta(T) = \mathbb{E}_T[T;\theta] - \tau(\theta) \qquad\qquad \mathrm{MSE}_\theta(T) = \mathbb{E}_T[(T-\tau(\theta))^2;\theta]$$

Suppose that $X_1, ..., X_n$ are a random sample from a $Poisson(\lambda)$ distribution. Find the maximum likelihood estimator of λ, and show that this estimator is unbiased. Also, find the maximum likelihood estimator of $\tau(\lambda) = e^{-\lambda} = \mathrm{P}_\theta[X=0]$, and find the approximate bias for this estimator using a Taylor expansion.

5. Suppose that $X_1, ..., X_n$ are a random sample from the probability distribution with pdf

$$f_X(x;\lambda,\eta) = \lambda e^{-\lambda(x-\eta)} \quad x > \eta$$

and zero otherwise. Find the maximum likelihood estimators of λ and η.

6. Suppose that $X_1, ..., X_n$ are a random sample from the probability distribution with pdf

$$f_X(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad\qquad x > 0.$$

Show that the sample mean $\overline{X}$ is an unbiased estimator of θ. Show also that, if random variable Y_1 is defined as $Y_1 = \min\{X_1, ..., X_n\}$ then random variable $Z = nY_1$ is also unbiased for θ.

7. Suppose that $X_1, ..., X_n$ are a random sample from a $Uniform(\theta - 1, \theta + 1)$ distribution. Show that the sample mean $\overline{X}$ is an unbiased estimator of θ. Let Y_1 and Y_n be the smallest and largest order statistics derived from $X_1, ..., X_n$. Show also that random variable $M = (Y_1 + Y_n)/2$ is an unbiased estimator of θ.

8. Suppose that $X_1, ..., X_n$ are a random sample from a $Gamma(2, \lambda)$ distribution.

(i) Find the maximum likelihood estimator of λ.

(ii) Find the maximum likelihood estimator, denoted T say, of $\tau = 1/\lambda$.

(iii) Find $\mathbb{E}_T[T;\lambda]$ and $\mathbb{E}_T\left[T^2;\lambda\right]$.

9. Consider the location family pdf with standard member the *Cauchy* distribution

$$f_X(x;\theta) = \frac{1}{\pi}\frac{1}{1 + (x-\theta)^2} \qquad -\infty < x < \infty$$

for location parameter $\theta \in \Theta \equiv \mathbb{R}$.

(a) Derive the *score equation* for θ defined for a random sample $X_1, \ldots, X_n$ from this pdf by

$$\frac{\partial \ell(\mathbf{x};\theta)}{\partial \theta} = 0$$

where $l(\mathbf{x};\theta) = \log \mathscr{L}(\mathbf{x};\theta) = \log f_{\mathbf{X}}(\mathbf{x};\theta)$

(b) Using a computer package, plot the log-likelihood function $\ell(\mathbf{x};\theta)$ for a suitable range of θ for the following observed x values:

7.36 5.14 3.71 3.15 6.00 6.38 1.34 6.73

and hence find the maximum likelihood (ML) estimate.

10. Carry out a simulation study to examine the sampling distribution of the maximum likelihood estimator $\widehat{\theta}(\mathbf{X})$ in the Cauchy location family example in the previous problem.

For example, in `R`:

- Produce $N = 5000$ simulated data sets of size $n = 8$, using a specific value of θ, and using the random number generation function `rcauchy`.
- For each simulated data set, use pointwise evaluation of the likelihood (or the function `optimize`) to evaluate the ML estimate in each case.
- Display using a histogram the distribution of the N stored ML estimates.

The sample median is an alternative estimator of θ. Repeat the computations above using this alternative estimator.