MATH 557 - EXERCISES 1

These exercises are not for assessment

1. Find sufficient statistics for random samples (that is, iid collections) of size n from the following distributions. Note that the sufficient statistics may be multidimensional, but must have dimension no greater than two.

   (a) The Beta density with parameters α and β

   $$f_X(x;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \qquad 0 < x < 1$$

   and zero otherwise, for $\alpha, \beta > 0$.

   (b) The pmf

   $$f_X(x;\theta) = \frac{(\log\theta)^x}{\theta x!} \qquad x = 0, 1, 2, \ldots$$

   and zero otherwise, for $\theta > 1$.

   (c) The Uniform density given by

   $$f_X(x;\theta) = \frac{1}{\theta} \qquad \theta < x < 2\theta$$

   and zero otherwise, for $\theta > 0$.

2. Suppose that $\mathbf{X} = (X_1, \ldots, X_n)^\top$ are a random sample from an $Exponential(\lambda)$ distribution,

   $$f_X(x;\lambda) = \lambda e^{-\lambda x} \qquad x > 0$$

   and zero otherwise, for $\lambda > 0$.

   (a) A sufficient statistic $T(\mathbf{X})$ is termed a *minimal sufficient statistic* if, for any other sufficient statistic, $T^*(\mathbf{X})$, $T(\mathbf{x})$ is a function of $T^*(\mathbf{x})$: for $\mathbf{x}, \mathbf{y} \in \mathbb{X}^n$, if $T(\mathbf{X})$ is minimal sufficient then

   $$T^*(\mathbf{x}) = T^*(\mathbf{y}) \quad \Longrightarrow \quad T(\mathbf{x}) = T(\mathbf{y})$$

   A necessary and sufficient condition for T to be a minimal sufficient statistic is that for two realizations $\mathbf{x}$ and $\mathbf{y}$,

   $$T(\mathbf{x}) = T(\mathbf{y}) \qquad \Longleftrightarrow \qquad \frac{f_X(\mathbf{x};\theta)}{f_X(\mathbf{y};\theta)} \text{ does not depend on } \theta.$$

   Using this result, derive a minimal sufficient statistic for λ in the Exponential model.

   (b) Suppose now that only the m smallest of the X_is are observed. Derive a sufficient statistic for λ based on this reduced sample of size m.

   Hint: Construct the joint density of the first m order statistics $X_{(1)}, \ldots, X_{(m)}$ by noting that

   $$X_{(m)} = x \qquad \Longleftrightarrow \qquad X_{(r)} > x, \textit{ for all } r > m.$$

3. Suppose that $\mathbf{X} = (X_1, \ldots, X_n)^\top$ are a random sample from a $Poisson(\theta)$ distribution. It can be shown that

   $$T(\mathbf{X}) = \sum_{i=1}^{n} X_i$$

   is a sufficient statistic for θ. Find the conditional mass function of $\mathbf{X}$ given that $T(\mathbf{X}) = t$.

4. Let $\mathbf{X} = (X_1, \ldots, X_n)^\top$ be a random vector with joint density in $\mathcal{F}_k = \{f_i(\mathbf{x}) : i = 0, \ldots, k\}$, so that $\mathcal{F}_k$ is parameterized by index $i \in \{0, \ldots, k\}$. Assume that the densities in $\mathcal{F}_k$ have common support.

   (a) Show that the statistic
   $$T(\mathbf{X}) = \left(\frac{f_1(\mathbf{X})}{f_0(\mathbf{X})}, \ldots, \frac{f_k(\mathbf{X})}{f_0(\mathbf{X})} \right)^\top$$
   is minimal sufficient for i.

   (b) Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be a family of densities with common support, and suppose that
   - $f_i \equiv f_{\theta_i} \in \mathcal{F}$, θ_i distinct, for $i = 0, \ldots, k$ and
   - $T(\mathbf{X})$ as defined in part (a) is sufficient for θ.

   Show that $T(\mathbf{X})$ is minimal sufficient for θ.

   (c) Show that 1-1 functions of minimal sufficient statistics are minimal sufficient statistics.

   (d) Use the results from parts (b) and (c) to show that the sample mean $\overline{X}$ is minimal sufficient for β if $\mathbf{X}$ is a random sample from an Exponential distribution with expectation $\beta > 0$.

5. Suppose that $X_1, \ldots, X_n$ are a random sample from a $Multinomial(3, \boldsymbol{\theta})$ distribution defined by the probabilities
$$\Pr[X_i = j] = \theta_j \qquad j = 1, 2, 3$$
and zero otherwise, where $0 < \theta_1, \theta_2, \theta_3 < 1$, and $\theta_1 + \theta_2 + \theta_3 = 1$.

   (a) Find a (possibly vector-valued) sufficient statistic for $\boldsymbol{\theta}$.

   (b) Find the (joint) pmf of the sufficient statistic.

   (c) Find the form of the maximum likelihood estimator for $\boldsymbol{\theta}$, that is, the vector of values of $\boldsymbol{\theta}$ for which the (log-)likelihood is maximized.