MATH 557 - ASSIGNMENT 4

To be handed in not later than 10pm, Tuesday 11th April 2017.
Please submit your solutions in pdf via myCourses.

1. In a Bayesian inference setting, suppose that along with exchangeability, the assumptions made about the scalar random variables $X_1, \ldots, X_n$ induce a Normal model for the 'likelihood' term in de Finetti's Theorem, that is, the X_i are conditionally independent given parameters (θ, σ^2),

$$X_i | \theta, \sigma^2 \sim Normal(\theta, \sigma^2) \qquad i = 1, \ldots, n.$$

Suppose further that the prior $\pi_0(\theta, \sigma^2)$ is specified to have the structure

$$\pi_0(\theta, \sigma^2) = \pi_0(\theta|\sigma^2)\pi_0(\sigma^2)$$

where

$$\begin{aligned} \pi_0(\sigma^2) &= \frac{(\beta/2)^{\alpha/2}}{\Gamma(\alpha/2)} \left(\frac{1}{\sigma^2}\right)^{\alpha/2+1} \exp\left\{-\frac{\beta}{2\sigma^2}\right\} \qquad InvGamma(\alpha/2, \beta/2) \\ \pi_0(\theta|\sigma^2) &= \left(\frac{\lambda}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\sigma^2}(\theta-\mu)^2\right\} \qquad Normal(\mu, \sigma^2/\lambda) \end{aligned}$$

where $\alpha, \beta, \lambda > 0$ and $\mu \in \mathbb{R}$ are fixed 'hyperparameters'. The prior assumption about σ^2 is that it is Inverse Gamma distributed, that is, that

$$\frac{1}{\sigma^2} \sim Gamma(\alpha/2, \beta/2).$$

   (a) Show that this prior is conjugate with the likelihood, that is, the posterior distribution $\pi_n(\theta, \sigma^2)$ has the same structure as the prior. 6 MARKS

   (b) Show that the expectation of the marginal posterior distribution for θ, $\pi_n(\theta)$ which is given by

$$\pi_n(\theta) = \int_0^\infty \pi_n(\theta, \sigma^2)\, d\sigma^2,$$

   can be expressed as a convex linear combination of the maximum likelihood estimator $\widehat{\theta}_n$, and the prior expectation μ. 4 MARKS

2. Consider the univariate natural Exponential Family distribution $f_{X|\theta}(x|\theta)$ specified by

$$f_{X|\theta}(x|\theta) = h(x)\exp\{\theta x - K(\theta)\}$$

for some function $h(x)$, for θ in some open interval, and where $K(\theta)$ is the *cumulant generating function*:

$$\int_{-\infty}^{\infty} h(x)\exp\{\theta x\}\, dx = e^{K(\theta)}.$$

Find the form of a prior distribution, $\pi_0(\theta)$, that is conjugate for a likelihood for a random sample of size n defined using $f_{X|\theta}(x|\theta)$, that is, a prior that leads to the posterior $\pi_n(\theta)$ that has the same functional form as $\pi_0(\theta)$, but has updated parameters. 4 MARKS

Question 3 is on page 2.

3. Suppose that 10 data points that are realizations of non-negative count random variables are recorded to have sum 36. If a $Poisson(\theta)$ model for the individual data is to be used conditional on parameter $\theta > 0$, construct

   (i) a Wald interval for θ with asymptotic confidence level 0.95;

   (ii) a likelihood-based interval for θ with asymptotic confidence level 0.95;

   (iii) a probability-symmetric Bayesian 'credible' interval for θ containing probability 0.95, that is, the interval (l, u) defined such that for posterior distribution $\pi_n(\theta)$

$$\int_0^l \pi_n(\theta)\, d\theta = \int_u^\infty \pi_n(\theta)\, d\theta = 0.025,$$

   if a prior distribution $\pi_0(\theta) \equiv Gamma(5, 2)$ is specified.

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Hint: for (ii) and (iii) a numerical solution is necessary. You may use any computing package to compute the intervals, but in `R` the functions `uniroot` and `qgamma` may be useful.