MATH 557 - ASSIGNMENT 3

To be handed in not later than 10pm, Monday 27th March 2017.
Please submit your solutions in pdf via myCourses.

The Delta Method gives a means of finding the asymptotic distribution of a function of an ML estimator. In its first-order form, it uses the Mean Value Theorem to assert that if, under the usual regularity conditions,

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} Z \sim Normal_k(\mathbf{0}_k, \Sigma(\theta_0))$$

and $g(.)$ is a $d \times 1$ function of θ, then

$$\sqrt{n}(g(\widehat{\theta}_n) - g(\theta_0)) \xrightarrow{d} \{\dot{g}(\theta_0)\}^\top Z \sim Normal_d(\mathbf{0}_d, \{\dot{g}(\theta_0)\}^\top \Sigma(\theta_0) \{\dot{g}(\theta_0)\})$$

provided

$$\dot{g}(\theta_0) = \left.\frac{\partial g(\theta)}{\partial \theta}\right|_{\theta=\theta_0} \qquad (k \times d)$$

exists and is a non-zero matrix. In the 1-d case, the result becomes

$$\sqrt{n}(g(\widehat{\theta}_n) - g(\theta_0)) \xrightarrow{d} Normal(0, \{\dot{g}(\theta_0)\}^2 \sigma_0^2)$$

provided $\dot{g}(\theta_0) \neq 0$, where $\sigma_0 \equiv \sigma(\theta_0)$.

This assignment focusses on the estimation of the 1-d Normal probability quantity

$$\tau_c(\theta_0) = \Pr_{\theta_0}[X \leq c]$$

for a fixed $c \in \mathbb{R}$, where $X \sim Normal(\theta_0, 1)$. Suppose that a random sample $X_1, \ldots, X_n$ distributed as $Normal(\theta_0, 1)$ is available.

(a) Find the maximum likelihood estimator of $\tau_c(\theta_0)$, denoted $\widehat{\tau}_n = \widehat{\tau}(\mathbf{X}) \equiv \widehat{\tau}(X_{1:n})$.

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(b) Using a Taylor expansion of the standard Normal cdf $\Phi(x)$, prove that that the bias of $\widehat{\tau}_n$ takes the form

$$-\frac{1}{2n}(c - \theta_0)\phi(c - \theta_0) + \mathrm{O}(n^{-2})$$

where $\phi(x)$ is the standard Normal pdf, and $\mathrm{O}(n^{-2})$ denotes terms that, as $n \longrightarrow \infty$, remain (absolutely) bounded even when multiplied by n^2.

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(c) Show that

$$\mathrm{Var}_{X_{1:n}}[\widehat{\tau}(X_{1:n}); \theta_0] = \frac{1}{n}\{\phi(c - \theta_0)\}^2 + \mathrm{O}(n^{-2}).$$

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(d) Find the asymptotic distribution (that is, the distribution as $n \longrightarrow \infty$) of

$$\sqrt{n}(\widehat{\tau}_n - \tau_c(\theta_0)).$$

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