MATH 557 - ASSIGNMENT 2

To be handed in not later than 10pm, Friday 24th February 2017.
Please submit your solutions in pdf via myCourses.

Consider the $Cauchy$ location family with pdf

$$f_X(x;\theta) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2} \qquad -\infty < x < \infty$$

for location parameter $\theta \in \Theta \equiv \mathbb{R}$. Data

$$7.36\ 5.14\ 3.71\ 3.15\ 6.00\ 6.38\ 1.34\ 6.73$$

are observed, and inference about θ is considered.

(a) Derive the recursive formula used to implement Newton's Method for finding $\widehat{\theta}_n$ for these data, and implement it, using the starting value $\widehat{\theta}(0)$ equal to the mean of the data points, to produce the values of $\widehat{\theta}^{(t)}$ for $t = 1, \ldots, 5$. For each recursive step, evaluate

$$\ell_n(\widehat{\theta}^{(t)}) \qquad \dot{\ell}_n(\widehat{\theta}^{(t)}) \qquad \ddot{\ell}_n(\widehat{\theta}^{(t)})$$

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(b) Consider the variables Y and Z with joint distribution specified by

$$\begin{aligned} Z &\sim Gamma(1/2, 1/2) \\ Y|Z = z &\sim Normal(\theta, 1/z) \end{aligned}$$

(i) Find the marginal distribution of Y, $f_Y(y;\theta)$. 2 MARKS

(ii) Find the conditional distribution of Z given $Y = y$, $f_{Z|Y}(z|y;\theta)$. 2 MARKS

(iii) Consider a random sample of size n from this distribution

$$(\mathbf{y}, \mathbf{z}) = \{(y_i, z_i), i = 1, \ldots, n\}.$$

Write down the "complete data" log-likelihood

$$\log f_{\mathbf{Y},\mathbf{Z}}(\mathbf{y}, \mathbf{z}; \theta) \tag{1}$$

2 MARKS

(iv) Compute the expectation of the random version of the function in (1), that is,

$$\log f_{\mathbf{Y},\mathbf{Z}}(\mathbf{Y}, \mathbf{Z}; \theta)$$

with respect to the conditional distribution $f_{\mathbf{Z}|\mathbf{Y}}(\mathbf{z}|\mathbf{y};\theta^{(*)})$, where $\theta^{(*)}$ is a possible value of θ. The expectation will be a function of θ, $\theta^{(*)}$ and $\mathbf{y}$ – denote this function $Q(\theta|\theta^{(*)})$. 3 MARKS

(v) Give an expression for the value of θ that maximizes $Q(\theta|\theta^{(*)})$. 2 MARKS

(vi) Use the expression in part (v) recursively, starting from $\widehat{\theta}^{(0)}$ equal to the mean of the data points, to produce the values of $\widehat{\theta}^{(t)}$ for $t = 1, \ldots, 5$, such that

$$\widehat{\theta}^{(t+1)} = \arg\max_{\theta} Q(\theta|\widehat{\theta}^{(t)}) \quad t = 1, 2, 3, 4, 5.$$

For each recursive step, evaluate

$$\ell_n(\widehat{\theta}^{(t)}) \qquad \dot{\ell}_n(\widehat{\theta}^{(t)})$$

where $\ell_n(\theta) = \log f_{\mathbf{Y}}(\mathbf{y};\theta)$, the likelihood computed from the marginal model in part (i).

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