

557: MATHEMATICAL STATISTICS II

NON-EXISTENCE OF UMP TESTS

In general, UMP tests cannot be found. This is often the case for tests at level α with two-sided alternative hypotheses of the form

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta \neq \theta_0 \end{aligned}$$

Consider $X_1, \dots, X_n \sim \text{Normal}(\theta, 1)$. If $\theta_1 \neq \theta_0$, by Neyman-Pearson, the UMP level α test $\mathcal{T}$ of

$$\begin{aligned} H_0 &: \theta = \theta_0 \\ H_1 &: \theta = \theta_1 \end{aligned}$$

must take the form

$$\frac{f_{\underline{X}|\theta}(\underline{x}|\theta_1)}{f_{\underline{X}|\theta}(\underline{x}|\theta_0)} = \frac{(2\pi)^{-n/2} \exp\{-n(\bar{x} - \theta_1)^2/2\} \exp\{-(n-1)s^2/2\}}{(2\pi)^{-n/2} \exp\{-n(\bar{x} - \theta_0)^2/2\} \exp\{-(n-1)s^2/2\}} = \frac{\exp\{-n(\bar{x} - \theta_1)^2/2\}}{\exp\{-n(\bar{x} - \theta_0)^2/2\}} > k$$

which is equivalent to

$$\bar{x}(\theta_1 - \theta_0) > k_1$$

Thus the nature of the UMP rejection region depends on the sign of $(\theta_1 - \theta_0)$;

$$\mathcal{R} \equiv \begin{cases} \{x : \bar{x} < c_n\} & \text{if } \theta_1 < \theta_0 \\ \{x : \bar{x} > c_n\} & \text{if } \theta_1 > \theta_0 \end{cases}$$

If $\theta_1 < \theta_0$; under H_0 , $\bar{X} \sim \text{Normal}(\theta_0, n^{-1})$, so the UMP test $\mathcal{T}_1$ of the simple hypotheses has rejection region defined by $\bar{x} < c_{n1}$ where

$$c_{n1} = \frac{\Phi^{-1}(\alpha)}{\sqrt{n}} + \theta_0$$

and power function

$$\beta_1(\theta) = \Pr[\bar{X} < c_{n1} | \theta] = \Phi(\Phi^{-1}(\alpha) + \sqrt{n}(\theta_0 - \theta))$$

where Φ and Φ^{-1} are the cdf and inverse cdf for the standard normal distribution. $\beta_1(\theta)$ is a decreasing function of θ .

If $\theta_1 > \theta_0$, the UMP test $\mathcal{T}_2$ of the simple hypotheses has rejection region defined by $\bar{x} > c_{n2}$ where

$$c_{n2} = \frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}} + \theta_0$$

and power function

$$\beta_2(\theta) = \Pr[\bar{X} > c_{n2} | \theta] = 1 - \Phi(\Phi^{-1}(1 - \alpha) + \sqrt{n}(\theta_0 - \theta))$$

$\beta_2(\theta)$ is an increasing function of θ . Hence there is no UMP test at level α , as

$$\begin{aligned} \text{for } \theta < \theta_0 &: \beta_1(\theta) < \beta_2(\theta) \\ \text{for } \theta > \theta_0 &: \beta_1(\theta) > \beta_2(\theta) \end{aligned}$$

and, by the Neyman-Pearson Lemma, the UMP level α test must have rejection region identical to that of $\mathcal{T}$.

As a partial resolution, consider the test $\mathcal{T}_3$ of the two-sided alternative hypothesis with rejection region

$$\mathcal{R}_3 \equiv \{\underline{x} : \bar{x} < -c_{n3}\} \cup \{\underline{x} : \bar{x} > c_{n3}\}$$

where

$$c_{n3} = \frac{\Phi^{-1}(1 - \alpha/2)}{\sqrt{n}} + \theta_0$$

This test has power function

$$\beta_3(\theta) = \Phi(\Phi^{-1}(\alpha/2) + \sqrt{n}(\theta_0 - \theta)) + 1 - \Phi(\Phi^{-1}(1 - \alpha/2) + \sqrt{n}(\theta_0 - \theta))$$

so that

$$\beta_3(\theta_0) = \alpha < \beta(\theta) \quad \forall \theta \neq \theta_0$$

so that the test is a level α test that is **unbiased**. In fact this test is UMP in the class of level α tests that are unbiased.

The figure below depicts the three power functions:

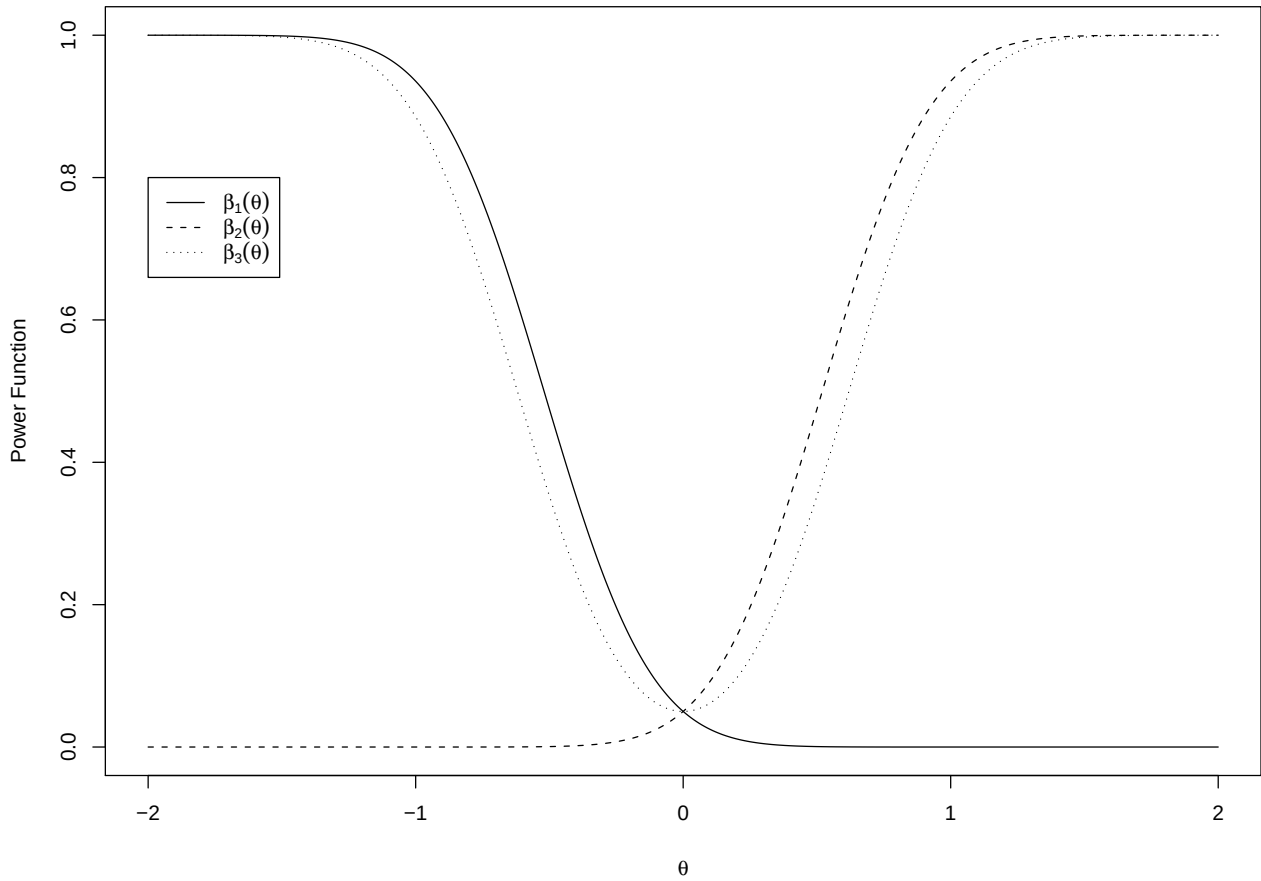


Figure 1: Power functions $\beta_1(\theta)$, $\beta_2(\theta)$ and $\beta_3(\theta)$ when $\theta_0 = 0$.