

557: MATHEMATICAL STATISTICS II

LIKELIHOOD, SUFFICIENCY AND CONDITIONALITY

Define a **statistical experiment**, E , as a triple

$$E \equiv \{ \underline{X}, \underline{\theta}, \{f_{X|\underline{\theta}}(\underline{x}|\underline{\theta})\} \}$$

comprising the sample $\underline{X}$, parameter $\underline{\theta}$ and family of distributions $\{f_{X|\underline{\theta}}(\underline{x}|\underline{\theta}) : \underline{\theta} \in \Theta\}$. Define also the **evidence**, $\text{Ev}(E, \underline{x})$, gathered about $\underline{\theta}$ in the experiment by observing $\underline{X} = \underline{x}$. We can restate earlier ideas using this notation as follows:

- **Formal Sufficiency Principle:** In experiment $E \equiv \{ \underline{X}, \underline{\theta}, \{f_{X|\underline{\theta}}(\underline{x}|\underline{\theta})\} \}$ suppose that $\underline{T}(\underline{X})$ is a sufficient statistic for $\underline{\theta}$. Then

$$\underline{T}(\underline{x}) = \underline{T}(\underline{y}) \implies \text{Ev}(E, \underline{x}) = \text{Ev}(E, \underline{y}).$$

- **Conditionality Principle:** Consider two experiments with common parameter $\underline{\theta}$

$$E_1 \equiv \{ \underline{X}_1, \underline{\theta}, \{f_1(\underline{x}|\underline{\theta})\} \} \quad E_2 \equiv \{ \underline{X}_2, \underline{\theta}, \{f_2(\underline{x}|\underline{\theta})\} \}.$$

Let Z be a random variable with $P[Z = 1] = P[Z = 2] = 1/2$ independent of $\underline{X}_1$ and $\underline{X}_2$, such that if $Z = z$ then experiment E_z is carried out, for $z = 1, 2$. Then the experiment carried out is

$$E^* \equiv \{ \underline{X}^*, \underline{\theta}, \{f^*(\underline{x}^*|\underline{\theta})\} \}$$

where $\underline{X}^* = (Z, \underline{X}_Z)^\top$, $\underline{x}^* = (z, \underline{x}_z)^\top$, and

$$f^*(\underline{x}^*|\underline{\theta}) = f^*((z, \underline{x}_z)|\underline{\theta}) = \frac{1}{2}f_z(\underline{x}_z|\underline{\theta}).$$

Then

$$\text{Ev}(E^*, \underline{x}^*) = \text{Ev}(E_z, \underline{x}_z)$$

that is, the information gathered about $\underline{\theta}$ depends only on the experiment carried out, and not the other experiment.

- **Formal Likelihood Principle:** Consider two experiments E_1 and E_2 with common parameter $\underline{\theta}$. Suppose that $\underline{x}_1^*$ and $\underline{x}_2^*$ are sample points from E_1 and E_2 such that

$$L(\underline{\theta}|\underline{x}_2^*) = C(\underline{x}_1^*, \underline{x}_2^*)L(\underline{\theta}|\underline{x}_1^*) \quad \forall \underline{\theta} \in \Theta,$$

where $C(\underline{x}_1^*, \underline{x}_2^*)$ is a constant function of $\underline{\theta}$. Then

$$\text{Ev}(E_1, \underline{x}_1^*) = \text{Ev}(E_2, \underline{x}_2^*)$$

that is, in experiment E , $\text{Ev}(E, \underline{x})$ depends on E only through the likelihood $L(\underline{\theta}|\underline{x})$.

These ideas are brought together in **Birnbaum's Theorem** (see Casella and Berger p 204), which states that the formal likelihood principle follows from the formal sufficiency principle and the conditionality principle, and *vice versa*.