

MATH 557 - MID-TERM EXAMINATION 2010

*Marks can be obtained by answering all questions.
50 marks available. Rescaling of the final mark may occur.*

1. (a) Suppose that $X \sim \text{Uniform}(-\theta, \theta)$ for some $\theta > 0$. Let $Y = |X|$. Is Y sufficient for θ ? Justify your answer.

4 Marks

- (b) Suppose that $X_1, \dots, X_n$ are a random sample from the $\text{Uniform}(-\theta, 2\theta)$ distribution. Find the maximum likelihood estimator of θ , $\hat{\theta}_{ML}(\underline{X})$. Is $\hat{\theta}_{ML}(\underline{X})$ sufficient for θ ? Justify your answer.

6 Marks

2. (a) Show that the family $\text{Normal}(0, \theta)$ for $0 < \theta < \infty$ is not complete.

4 Marks

- (b) Consider a random sample $X_1, \dots, X_n$ from the pdf

$$f_{X|\theta_1, \theta_2}(x|\theta_1, \theta_2) = \frac{1}{\theta_2} \exp\{-(x - \theta_1)/\theta_2\} \mathbb{1}_{(\theta_1, \infty)}(x)$$

for $-\infty < \theta_1 < \infty$, $\theta_2 > 0$. Find the (two-dimensional) minimal sufficient statistic for the vector (θ_1, θ_2) .

6 Marks

3. Suppose that $X_1, \dots, X_n$ are a random sample from a $\text{Normal}(\theta, \sigma^2)$ distribution. Suppose σ^2 is known.

- (a) Using a conjugate prior for θ , find the Bayesian posterior distribution based on the observed data $x_1, \dots, x_n$.

5 Marks

- (b) Show that the Bayes estimator for θ under squared-error loss takes the form

$$\hat{\theta}_B(\underline{X}) = w\hat{\theta}_{ML}(\underline{X}) + (1 - w)\mu$$

where $0 < w < 1$, $\hat{\theta}_{ML}(\underline{X})$ is the maximum likelihood estimator of θ , and μ is a parameter in the conjugate prior. Note that w may be a function of n and the prior parameters.

5 Marks

4. Suppose that $X_1, \dots, X_n$ is a random sample from a $Poisson(\lambda)$ distribution.

(a) Find the maximum likelihood estimator of $\tau(\lambda) = \{P_\lambda[X_1 = 0]\}^2$

4 Marks

(b) Consider the estimator of $\tau(\lambda)$, $\hat{\tau}_1(\underline{X}) = (-1)^{X_1}$. Show that $\hat{\tau}_1(\underline{X})$ is unbiased for τ .

4 Marks

(c) Comment on $\hat{\tau}_1(\underline{X})$ as an estimator of τ .

2 Marks

5. (a) Suppose that $X_1, \dots, X_n$ is a random sample from a $Normal(\theta, 1)$ distribution. Consider parameter $\phi(\theta) = P_\theta[X \leq c]$ for some fixed c . Find an unbiased estimator for $\phi(\theta)$.

5 Marks

(b) Suppose that $X_1, \dots, X_n$ is a random sample from a continuous distribution with cdf $F_X(x)$. Let $\theta = P[X \leq a]$ for some fixed a . Show that the estimator

$$T(\underline{X}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, a]}(X_i)$$

is unbiased for θ , and that it achieves the Cramér-Rao variance bound, that is

$$\text{Var}_{f_{T|\theta}}[T] = \frac{1}{n\mathcal{I}(\theta)}$$

where $\mathcal{I}(\theta)$ is the Fisher information for θ .

Hint: Consider the random variables $Y_i = \mathbb{1}_{(-\infty, a]}(X_i)$, $i = 1, \dots, n$ and their distribution.

5 Marks