

MATH 557 - MID-TERM 2010 - SOLUTIONS

1. (a) Y is sufficient for θ , as, given $Y = y$, the conditional distribution of X is concentrated on $\{-y, y\}$ irrespective of the value of θ .

4 Marks

- (b) The likelihood is

$$L(\theta|\underline{x}) = f_{\underline{X}|\theta}(\underline{x}|\theta) = \frac{\prod_{i=1}^n \mathbb{1}_{(-\theta, 2\theta)}(x_i)}{3\theta} = \frac{\mathbb{1}_{(T(\underline{x}), \infty)}(\theta)}{3^n \theta^n}$$

where $T(\underline{x}) = \max\{-x_{(1)}, x_{(n)}/2\}$. This function is monotonic decreasing in θ , so therefore $\hat{\theta}_{ML} = T(\underline{x})$. By the factorization theorem, $\hat{\theta}_{ML}(\underline{X})$ is sufficient.

6 Marks

2. (a) The family $Normal(0, \theta)$ for $0 < \theta < \infty$ is not complete, as if $g(t) = t$, then

$$E_{f_{T|\theta}}[g(T)] = E_{f_{T|\theta}}[T] = 0$$

for all θ , even though $g(T)$ is not zero with probability 1.

4 Marks

- (b) The joint pdf is

$$\begin{aligned} f_{\underline{X}|\theta_1, \theta_2}(\underline{x}|\theta_1, \theta_2) &= \frac{\prod_{i=1}^n \mathbb{1}_{(\theta_1, \infty)}(x_i)}{\theta_2^n} \exp \left\{ -\frac{1}{\theta_2} \sum_{i=1}^n (x_i - \theta_1) \right\} \\ &= \frac{\prod_{i=1}^n \mathbb{1}_{(x_{(1)}, \infty)}(\theta_1)}{\theta_2^n} \exp \left\{ -\frac{\sum_{i=1}^n x_i}{\theta_2} - n\theta_1/\theta_2 \right\} \end{aligned}$$

Therefore, by the theorem, as the ratio $f_{\underline{X}|\theta_1, \theta_2}(\underline{x}|\theta_1, \theta_2)/f_{\underline{X}|\theta_1, \theta_2}(\underline{y}|\theta_1, \theta_2)$ is independent of (θ_1, θ_2) for all possible values of the parameters if and only if

$$x_{(1)} = y_{(1)} \quad \text{and} \quad \sum_{i=1}^n x_i = \sum_{i=1}^n y_i$$

it follows that $\underline{T}(\underline{X}) = (X_{(1)}, \sum_{i=1}^n X_i)$ is a minimal sufficient statistic.

6 Marks

3. (a) The conjugate prior for θ is a Normal prior $Normal(\mu, \tau^2)$ say. In the exponent of the resulting posterior distribution, we have

$$\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \theta)^2 + \frac{1}{\tau^2} (\theta - \mu)^2 = \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{n}{\sigma^2} (\bar{x} - \theta)^2 + \frac{1}{\tau^2} (\theta - \mu)^2$$

and to see the form of the posterior distribution of θ , we need to complete the square in θ .

We have that

$$A(\theta - a)^2 + B(\theta - b)^2 = (A + B)(\theta - m)^2 + \frac{AB}{A + B}(a - b)^2$$

where

$$m = \frac{Aa + Bb}{A + B}$$

Here we have

$$A = \frac{n}{\sigma^2} \quad a = \bar{x} \quad B = \frac{1}{\tau^2} \quad b = .$$

Hence the term in the exponent is

$$\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \bar{x})^2 + \left[\frac{n}{\sigma^2} + \frac{1}{\tau^2} \right] (\theta - \mu^*)^2 + \frac{n/(\sigma^2 \tau^2)}{n/\sigma^2 + 1/\tau^2} (\bar{x} - \mu)^2$$

where

$$\mu^* = \frac{n\bar{x}/\sigma^2 + \mu/\tau^2}{n/\sigma^2 + 1/\tau^2} = \frac{n\tau^2\bar{x} + \mu\sigma^2}{n\tau^2 + \sigma^2} = w\bar{x} + (1 - w)\mu$$

where $w = n\tau^2/(n\tau^2 + \sigma^2)$. The posterior distribution for θ is therefore

$$\pi_{\theta|\underline{X}}(\theta|\underline{x}) \propto \exp \left\{ -\frac{(n\tau^2 + \sigma^2)}{2\sigma^2\tau^2} (\theta - \mu^*)^2 \right\}$$

so $\theta|\underline{x} \sim \text{Normal}(\mu^*, \sigma^2\tau^2/(n\tau^2 + \sigma^2))$.

5 Marks

(b) The Bayes estimator is the posterior mean, and hence

$$\hat{\theta}_B(\underline{X}) = w\bar{X} + (1 - w)\mu.$$

given the above derivation.

5 Marks

4. (a) By standard arguments, the ML estimator is $\bar{X}$, and as $\tau(\lambda) = e^{-2\lambda}$, by invariance we have that

$$\hat{\tau}_{ML}(\underline{X}) = \exp\{-2\bar{X}\}.$$

4 Marks

(b) We have that

$$E[\hat{\tau}_1(\underline{X})] = E[(-1)^{X_1}] = \sum_{x=0}^{\infty} (-1)^x \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(-\lambda)^x}{x!} = e^{-2\lambda}$$

4 Marks

- (c) It is unbiased, but only takes values in the set $\{-1, 1\}$, whatever the values of λ and the data. Clearly, -1 is not sensible as an estimate of λ , and this estimator is not reasonable, despite its unbiasedness.

2 Marks

5. (a) Consider

$$T(\underline{X}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, c]}(X_i).$$

Then

$$\mathbb{E}[T(\underline{X})] = \frac{1}{n} \sum_{i=1}^n P_\theta[X_i \leq c] = \frac{n\phi(\theta)}{n} = \phi(\theta)$$

5 Marks

(b) We have for the defined $Y_1, \dots, Y_n$ that

$$f_{Y_i|\theta}(y|\theta) = \theta^y(1-\theta)^{1-y}$$

and hence

$$\sum_{i=1}^n \mathbb{1}_{(-\infty, a]}(X_i) = \sum_{i=1}^n Y_i \sim \text{Binomial}(n, \theta).$$

For this model, if $T_0(\underline{y}) = \sum_{i=1}^n y_i$,

$$S_\theta(\underline{y}) = \frac{\partial \log L(\theta|\underline{y})}{\partial \theta} = \frac{\partial}{\partial \theta} \left\{ T_0(\underline{y}) \log \theta + (n - T_0(\underline{y})) \log(1 - \theta) \right\} = \frac{T_0(\underline{y})}{\theta} - \frac{n - T_0(\underline{y})}{1 - \theta} = \frac{T_0(\underline{y}) - n\theta}{\theta(1 - \theta)}$$

The n -data Fisher information is

$$\begin{aligned} \mathcal{I}_n(\theta) = \text{Var}_{\underline{Y}|\theta}[S_\theta(\underline{Y})] &= \frac{1}{\theta^2(1-\theta)^2} \mathbb{E}_{\underline{X}|\theta}[(T_0(\underline{X}) - n\theta)^2] \\ &= \frac{1}{\theta^2(1-\theta)^2} \text{Var}_{\underline{X}|\theta}[T_0(\underline{X})] \\ &= \frac{n\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{n}{\theta(1-\theta)} \end{aligned}$$

But

$$\text{Var}_{T|\theta}[T(\underline{X})] = \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n} = \frac{1}{\mathcal{I}_n(\theta)} = \frac{1}{n\mathcal{I}(\theta)}$$

where $\mathcal{I}(\theta)$ is the unit Fisher information.

5 Marks