

MATH 557 - EXERCISES 4 SOLUTIONS

1 (a) To find the UMP test, consider

$$\begin{aligned} H_0 &: \theta = 1 \\ H_1 &: \theta = \theta_1 \end{aligned}$$

for $\theta_1 > 1$. By Neyman-Pearson, the rejection region is constructed by looking at

$$\frac{f_{\underline{X}|\theta}(\underline{x}|\theta_1)}{f_{\underline{X}|\theta}(\underline{x}|1)} = \frac{\prod_{i=1}^n \theta_1 (1 - x_i)^{\theta_1 - 1}}{1} = \theta_1^n \{T(\underline{x})\}^{\theta_1 - 1}$$

where $T(\underline{x}) = \prod_{i=1}^n (1 - x_i)$. Hence the rejection region is defined by

$$\theta_1^n \{T(\underline{x})\}^{\theta_1 - 1} > k \quad \text{or equivalently} \quad T(\underline{x}) > k_1$$

where the requirement

$$\Pr[T(\underline{X}) \in \mathcal{R}_T | \theta = 1] = \Pr[T(\underline{X}) > k_1 | \theta = 1] = \alpha$$

determines k_1 for any α . To simplify further

$$\prod_{i=1}^n (1 - X_i) > k_1 \quad \Longleftrightarrow \quad -\sum_{i=1}^n \log(1 - X_i) < -\log k_1 = c$$

say. Now, if $\theta = 1$, the data are uniformly distributed on $(0,1)$. Also, if $X \sim \text{Uniform}(0, 1)$, then $1 - X \sim \text{Uniform}(0, 1)$, and

$$-\log(1 - X) \sim \text{Exponential}(1)$$

Therefore the critical region is defined by

$$\Pr[T(\underline{X}) > k_1 | \theta = 1] = \Pr[V < c | \theta = 1] = \alpha$$

where

$$V = -\log T(\underline{X}) = -\sum_{i=1}^n \log(1 - X_i) \sim \text{Gamma}(n, 1).$$

Thus c is the α quantile of the $\text{Gamma}(n, 1)$ distribution. This is the UMP test for any $\theta_1 > 1$, so it is the UMP test for the required hypotheses.

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(b) Under H_1 , the ML estimate of θ is

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^+}{\operatorname{argmax}} \theta^n \{T(\underline{x})\}^{\theta - 1} = -\frac{n}{\log T(\underline{x})} = -\frac{n}{\sum_{i=1}^n \log(1 - X_i)} = -\frac{n}{\log T(\underline{x})}$$

Thus the LRT is based on the rejection region $\mathcal{R}_{\underline{X}}$ defined by

$$\lambda_{\underline{X}}(\underline{x}) = \frac{L(1|\underline{x})}{L(\hat{\theta}|\underline{x})} = \frac{1}{\hat{\theta}^n \{T(\underline{x})\}^{\hat{\theta} - 1}} \leq k$$

which is equivalent to

$$n \log \hat{\theta} + (\hat{\theta} - 1) \log T(\underline{x}) \geq -\log k$$

or

$$-n \log(-\log T(\underline{x})) - \log T(\underline{x}) \geq -\log k - n \log n + n$$

which may be written

$$-n \log V + V \geq c$$

where $V \sim \text{Gamma}(n, 1)$ as above. To solve this for c requires numerical steps.

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2 Can use the Karlin-Rubin theorem in both cases.

(a) The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underline{x}) = \frac{f_{\underline{X}|\theta}(\underline{x}|\theta_2)}{f_{\underline{X}|\theta}(\underline{x}|\theta_1)} = \frac{\theta_1^n}{\theta_2^n} \exp \left\{ T(\underline{x}) \left(\frac{1}{\theta_1} - \frac{1}{\theta_2} \right) \right\}$$

which is an increasing function of $T(\underline{x}) = \sum_{i=1}^n x_i$. Thus the rejection region takes the form

$$\mathcal{R} \equiv \left\{ \underline{x} : T(\underline{x}) = \sum_{i=1}^n x_i > t_0 \right\}$$

To find t_0 , we need to solve

$$\Pr[T(\underline{X}) > t_0 \mid \theta_0] = \alpha.$$

Here $T(\underline{X}) \sim \text{Gamma}(n, 1/\theta)$, so t_0 is the $1 - \alpha$ quantile of this distribution.

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(b) The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underline{x}) = \frac{f_{\underline{X}|\theta}(\underline{x}|\theta_2)}{f_{\underline{X}|\theta}(\underline{x}|\theta_1)} = \frac{\theta_1^{n/2}}{\theta_2^{n/2}} \exp \left\{ \frac{T(\underline{x})}{2} \left(\frac{1}{\theta_1} - \frac{1}{\theta_2} \right) \right\}$$

which is an increasing function of $T(\underline{x}) = \sum_{i=1}^n (x_i - 1)^2$. Thus the rejection region takes the form

$$\mathcal{R} \equiv \left\{ \underline{x} : T(\underline{x}) = \sum_{i=1}^n x_i > t_0 \right\}$$

To find t_0 , we need to solve

$$\Pr[T(\underline{X}) > t_0 \mid \theta_0] = \alpha.$$

Here under the assumption $\theta = \theta_0$,

$$\frac{T(\underline{X})}{\theta_0} \sim \chi_n^2 \equiv \text{Gamma}(n/2, 1/2)$$

so

$$\Pr[T(\underline{X}) > t_0 \mid \theta_0] = \Pr[T(\underline{X})/\theta_0 > t_0/\theta_0 \mid \theta_0] = \alpha.$$

implies that $t_0 = \theta_0 q_{n, 1-\alpha}$, where $q_{n, 1-\alpha}$ is the $1 - \alpha$ quantile of the Chisquared distribution with n degrees of freedom.

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3 Again using the Karlin-Rubin Theorem: The likelihood ratio for $\theta_1 < \theta_2$ for this model is

$$\lambda(\underline{x}) = \frac{f_{\underline{X}|\theta}(\underline{x}|\theta_2)}{f_{\underline{X}|\theta}(\underline{x}|\theta_1)} = \left(\frac{\theta_2}{\theta_1}\right)^{T(\underline{x})} \exp\{-n(\theta_2 - \theta_1)\}$$

where $T(\underline{x}) = \sum_{i=1}^n x_i$. In this case, under $\theta = 2$,

$$T(\underline{X}) = \sum_{i=1}^n X_i \sim \text{Poisson}(2n)$$

Thus the distribution of $T(\underline{X})$ is discrete. A randomized test takes the form

$$\phi_{\mathcal{R}}^*(\underline{x}) = \begin{cases} 1 & T(\underline{x}) > c \\ \gamma & T(\underline{x}) = c \\ 0 & T(\underline{x}) \leq c \end{cases}$$

where c is the largest integer such that

$$\Pr[T(\underline{X}) > c] \leq 0.05$$

and γ is selected so that

$$\Pr[T(\underline{X}) > c] + \gamma \Pr[T(\underline{X}) = c] = 0.05$$

In the example, $n = 6$, and $T(\underline{x}) = 18$, and by calculation $c = 18$

$$\Pr[T(\underline{X}) > 18] = 0.0374 \quad \Pr[T(\underline{X}) = 18] = 0.0255$$

so that

$$\gamma = \frac{0.05 - \Pr[T(\underline{X}) > 18]}{\Pr[T(\underline{X}) = 18]} = \frac{0.05 - 0.0374}{0.0255} = 0.494$$

In this case, the hypothesis is rejected with probability $\gamma = 0.494$ as $T(\underline{x}) = 18$.

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