

MATH 557 - EXERCISES 3 SOLUTIONS

1 Let $X \sim \text{Binomial}(n, \theta)$ for $0 < \theta < 1$.

(a) By standard results for expectations

$$E_{f_{T|\theta}}[T|\theta] = \frac{1}{n} E_{f_{X|\theta}}[X|\theta] = \frac{n\theta}{n} = \theta$$

(b) Suppose that $U(X)$ is an unbiased estimator of $\tau(\theta)$, so that

$$E_{f_{U|\theta}}[U|\theta] = \sum_{x=0}^n U(x) \binom{n}{x} \theta^x (1-\theta)^{n-x} = \frac{1}{\theta}$$

Clearly $U(x)$ must be finite on $0 \leq x \leq n$, otherwise the expectation is not finite, in which case $U(X)$ is biased for any θ . But also, for $0 < \theta < 1$

$$\sum_{x=0}^n U(x) \binom{n}{x} \theta^x (1-\theta)^{n-x} \leq \sum_{x=0}^n U(x) \binom{n}{x} = M(n) < \infty$$

so therefore the expectation of $U(X)$ is bounded above. Therefore, if $\theta < 1/M(n)$, the expectation cannot attain $1/\theta$, so the estimator is not unbiased; no unbiased estimator exists.

2 (a) The ML estimator of θ is $M(\underline{X}) = X_{(n)} = \max\{X_1, \dots, X_n\}$ which has pdf

$$f_{M|\theta}(m|\theta) = \frac{nm^{n-1}}{\theta^n} \quad 0 < m < \theta$$

which has expectation

$$E_{f_{M|\theta}}[M|\theta] = \int_0^\theta \frac{nm^n}{\theta^n} dm = \frac{n}{n+1} \theta$$

so therefore $T = (n+1)M/n$ is unbiased for θ .

(b) The variance/MSE of $T(\underline{X})$ is

$$\begin{aligned} \text{Var}_{f_{T|\theta}}[T|\theta] &= E_{f_{T|\theta}}[T^2|\theta] - \theta^2 = \left(\frac{n+1}{n}\right)^2 E_{f_{M|\theta}}[M^2|\theta] - \theta^2 \\ &= \left(\frac{n+1}{n}\right)^2 \int_0^\theta m^2 \frac{nm^{n-1}}{\theta^n} dm - \theta^2 \\ &= \left(\frac{n+1}{n}\right)^2 \left(\frac{n+2}{n}\right) \theta^2 - \theta^2 \end{aligned}$$

(c) The Cramér-Rao bound is

$$B(\theta) = \frac{\left(\frac{d}{d\theta} \{E_{f_{T|\theta}}[T]\}\right)^2}{E_{f_{\underline{X}|\theta}}[S(\underline{X}; \theta)^2]} = \frac{(\dot{\tau}(\theta))^2}{E_{f_{\underline{X}|\theta}}[n/\theta^2]} = \frac{1}{n/\theta^2} = \frac{\theta^2}{n}$$

3 (a) The likelihood, log-likelihood and derivative are

$$L(\theta|\underline{x}) = \theta^n \left(\prod_{i=1}^n x_i \right)^{\theta-1} \quad \theta > 0$$

$$l(\theta|\underline{x}) = n \log \theta + (\theta - 1) \sum_{i=1}^n \log x_i$$

$$\dot{l}(\theta|\underline{x}) = \frac{n}{\theta} + \sum_{i=1}^n \log x_i$$

and hence

$$\hat{\theta}_n(\underline{X}) = -\frac{n}{\sum_{i=1}^n \log X_i} = \frac{n}{T(\underline{X})} \quad \text{say.}$$

(b) If $X \sim \text{Beta}(\theta, 1)$ then if $Y = -\log X$, we have

$$F_{Y|\theta}(y|\theta) = \Pr[Y \leq y|\theta] = \Pr[-\log X \leq y|\theta] = \Pr[X \geq e^{-y}|\theta] = 1 - F_{X|\theta}(e^{-y}|\theta)$$

$$\therefore f_{Y|\theta}(y|\theta) = e^{-y} f_{X|\theta}(e^{-y}|\theta) = e^{-y} \theta (e^{-y})^{\theta-1} = \theta e^{-\theta y} \quad y > 0$$

so $Y \sim \text{Exponential}(\theta)$ and $T(\underline{X}) \sim \text{Gamma}(n, \theta)$. Now for $r = 1, 2, \dots$

$$\mathbb{E}_{f_{T|\theta}}[T^{-r}] = \int_0^\infty \frac{1}{t^r} \frac{\theta^n}{\Gamma(n)} t^{n-1} e^{-\theta t} dt = \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-r)}{\theta^{n-r}} = \frac{\theta^r}{(n-1)(n-2)\dots(n-r+1)}.$$

Hence

$$\mathbb{E}_{f_{T|\theta}}[T^{-1}] = \frac{\theta}{(n-1)} \quad \mathbb{E}_{f_{T|\theta}}[T^{-2}] = \frac{\theta^2}{(n-1)(n-2)}$$

so that

$$\text{Var}_{f_{T|\theta}}[T^{-1}] = \frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} = \frac{\theta^2}{(n-1)^2(n-2)}$$

Thus the expectation and variance of $\hat{\theta}_n(\underline{X})$ are

$$\frac{n\theta}{(n-1)} \quad \frac{n^2\theta^2}{(n-1)^2(n-2)}$$

so $\hat{\theta}_n(\underline{X})$ is not unbiased for θ .

(c) Writing $T_1(\underline{X}) = \hat{\theta}_n(\underline{X})$, the Cramér-Rao bound is

$$\begin{aligned} B(\theta) &= \frac{\left(\frac{d}{d\theta} \left\{ \mathbb{E}_{f_{T_1|\theta}}[T_1] \right\} \right)^2}{\mathbb{E}_{f_{\underline{X}|\theta}}[S(\underline{X}; \theta)^2]} = \frac{(n/(n-1))^2}{\mathbb{E}_{f_{T|\theta}}[(n/\theta - T)^2]} = \frac{\left(\frac{n}{n-1} \right)^2}{\frac{n^2}{\theta^2} - 2\frac{n}{\theta} \mathbb{E}_{f_{T|\theta}}[T] + \mathbb{E}_{f_{T|\theta}}[T^2]} \\ &= \frac{\left(\frac{n}{n-1} \right)^2}{\frac{n^2}{\theta^2} - 2\frac{n}{\theta} \frac{n}{n-1} + \frac{n(n+1)}{\theta^2}} = \frac{n\theta^2}{(n-1)^2} \end{aligned}$$

Thus the Cramér-Rao bound is not met here despite the fact that this is an Exponential Family distribution. The Exponential Family does yield unbiased estimators of parameters, but only in specific parameterizations relating to the natural parameterization. In this case, we have

$$f_{X|\theta}(x|\theta) = \theta \exp\{(\theta - 1) \log x\}$$

so that, by properties of the score function

$$E_{f_{X|\theta}}[-\log X|\theta] = \frac{1}{\theta} = \tau(\theta)$$

and the result yields that

$$T_2(\underline{X}) = -\frac{1}{n} \sum_{i=1}^n \log X_i$$

is an unbiased estimator of $\tau(\theta)$. As

$$T_2(\underline{X}) \sim \text{Gamma}(n, n\theta)$$

we have that

$$\text{Var}_{f_{T_2|\theta}}[T_2|\theta] = \frac{1}{n\theta^2}$$

for which the Cramér-Rao bound is

$$B(\theta) = \frac{\left(\frac{d}{d\theta} \{E_{f_{T_2|\theta}}[T_2]\}\right)^2}{E_{f_{\underline{X}|\theta}}[S(\underline{X}; \theta)^2]} = \frac{(\dot{\tau}(\theta))^2}{E_{f_{T_2|\theta}}[(n/\theta - nT_2)^2]} = \frac{1/\theta^4}{n^2 \text{Var}_{f_{T_2|\theta}}[T_2]} = \frac{1}{n\theta^2}$$

so the bound is attained.

4 The likelihood, log-likelihood and derivative are

$$L(\theta|\underline{x}) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left\{-\frac{1}{2\theta} \sum_{i=1}^n x_i^2\right\} \theta > 0$$

$$l(\theta|\underline{x}) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \theta - \frac{1}{2\theta} \sum_{i=1}^n x_i^2$$

$$\dot{l}(\theta|\underline{x}) = -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n x_i^2$$

and hence

$$\hat{\theta}_n(\underline{X}) = \frac{1}{n} \sum_{i=1}^n X_i^2.$$

As $X_i^2/\theta \sim \chi_1^2 \equiv \text{Gamma}(1/2, 1/2)$, so that

$$\hat{\theta}_n(\underline{X}) = \frac{\theta}{n} \sum_{i=1}^n (X_i^2/\theta) \sim \text{Gamma}(n/2, n/2\theta)$$

so $\hat{\theta}_n(\underline{X})$ has expectation θ and variance $2\theta^2/n$. In this problem, the Cramér-Rao bound is

$$\begin{aligned} B(\theta) &= \frac{\left(\frac{d}{d\theta} \left\{ \mathbb{E}_{f_{T_2|\theta}}[T_2] \right\} \right)^2}{\mathbb{E}_{f_{\underline{X}|\theta}}[S(\underline{X}; \theta)^2]} = \frac{1}{\mathbb{E}_{f_{T|\theta}}[-\Psi(\underline{X}; \theta)]} = \frac{1}{\mathbb{E}_{f_{T|\theta}} \left[\frac{n}{2\theta^2} + \frac{1}{\theta^3} \sum_{i=1}^n X_i^2 \right]} \\ &= \frac{1}{-\frac{n}{2\theta^2} + \frac{n}{\theta^2}} = \frac{2\theta^2}{n} \end{aligned}$$

so the bound is attained by the ML estimator, and $\hat{\theta}_n(\underline{X})$ is the best unbiased estimator.

5 (a) We have that

$$\begin{aligned} \mathcal{I}(\theta) &= \mathbb{E}_{f_{\underline{X}|\theta}}[S(\underline{X}; \theta)^2] = \int \left\{ \frac{\partial}{\partial \theta} \log f_{X|\theta}(x|\theta) \right\}^2 f_{X|\theta}(x|\theta) dx \\ &= \int \left\{ \frac{\dot{f}(x-\theta)}{f(x-\theta)} \right\}^2 f(x-\theta) dx \\ &= \int \left\{ \frac{\dot{f}(x)}{f(x)} \right\}^2 f(x) dx \end{aligned}$$

after making the change of variables $x \rightarrow x - \theta$. Hence $\mathcal{I}(\theta)$ is constant in θ .

(b) For the Double Exponential model

$$\begin{aligned} \mathcal{I}(\theta) &= \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \theta} \log f_{X|\theta}(x|\theta) \right\}^2 f_{X|\theta}(x|\theta) dx = \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \theta} - |x - \theta| \right\}^2 \frac{1}{2} e^{-|x-\theta|} dx \\ &= \int_{-\infty}^{\theta} \left\{ \frac{\partial}{\partial \theta} (x - \theta) \right\}^2 \frac{1}{2} e^{(x-\theta)} dx + \int_{\theta}^{\infty} \left\{ \frac{\partial}{\partial \theta} - (x - \theta) \right\}^2 \frac{1}{2} e^{-(x-\theta)} dx \\ &= \int_{-\infty}^{\theta} \frac{1}{2} e^{(x-\theta)} dx + \int_{\theta}^{\infty} \frac{1}{2} e^{-(x-\theta)} dx = 1 \end{aligned}$$

Note that the lack of differentiability of the pdf at θ does not render this computation invalid, as this is a single point in a continuous distribution and hence has probability zero, that is, the function is differentiable *almost everywhere*.

(c) For the Logistic model

$$\begin{aligned} \mathcal{I}(\theta) &= \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial \theta} \log f_{X|\theta}(x|\theta) \right\}^2 f_{X|\theta}(x|\theta) dx = \int_{-\infty}^{\infty} \left\{ -\frac{\partial^2}{\partial \theta^2} \log f_{X|\theta}(x|\theta) \right\} f_{X|\theta}(x|\theta) dx \\ &= \int_{-\infty}^{\infty} \frac{2}{(1 + e^{-(x-\theta)})^2} \frac{e^{-(x-\theta)}}{(1 + e^{-(x-\theta)})^2} dx \\ &= \int_{-\infty}^{\infty} \frac{2e^{-x}}{(1 + e^{-x})^4} dx \quad (x - \theta \rightarrow x) \\ &= \left[\frac{2}{3(1 + e^{-x})^3} \right]_{-\infty}^{\infty} = \frac{2}{3} \end{aligned}$$

6 For this pdf, the corresponding expectation is

$$E_{f_{X|\theta}}[X|\theta] = \int_0^\infty x \frac{3\theta^3}{(x+\theta)^4} dx = \left[\frac{-x\theta^3}{(x+\theta)^3} \right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^3} dx = 0 + \left[-\frac{\theta^3}{2(x+\theta)^2} \right]_0^\infty = \frac{\theta}{2}$$

so therefore

$$T(\underline{X}) = \frac{2}{n} \sum_{i=1}^n X_i$$

is an unbiased estimator of θ . By a similar calculation

$$\begin{aligned} E_{f_{X|\theta}}[X^2|\theta] &= \int_0^\infty x^2 \frac{3\theta^3}{(x+\theta)^4} dx = \left[\frac{-x^2\theta^3}{(x+\theta)^3} \right]_0^\infty + \int_0^\infty \frac{2x\theta^3}{(x+\theta)^3} dx \\ &= 0 + \left[-\frac{x\theta^3}{(x+\theta)^2} \right]_0^\infty + \int_0^\infty \frac{\theta^3}{(x+\theta)^2} dx = 0 + \left[-\frac{\theta^3}{(x+\theta)} \right]_0^\infty = \theta^2 \end{aligned}$$

so therefore

$$\text{Var}_{f_{X|\theta}}[X|\theta] = \theta^2 - \frac{\theta^2}{4} = \frac{3\theta^2}{4} \quad \therefore \quad \text{Var}_{f_{T|\theta}}[T|\theta] = \frac{3\theta^2}{n}$$

7 There were some typos in the original version of the question, with T and S swapped.

In each case, we need $E[T|S]$

(a) The variance of T is λ , but by symmetry

$$U = E[T|S] = E[X_1|S] = \frac{S}{n} = \frac{1}{n} \sum_{i=1}^n X_i$$

which has variance λ/n .

(b) We have, as T is binary,

$$E[T] = E[T^2] = \Pr[X_1 = 0] = e^{-\lambda} \quad \therefore \quad \text{Var}_{f_{T|\theta}}[T] = e^{-\lambda}(1 - e^{-\lambda})$$

and

$$U = E[T|S] = E[\mathbb{1}_{\{0\}}(X_1)|S] = \Pr[X_1 = 0|S]$$

The conditional density of X_1 given S is, by independence of $X_1, \dots, X_n$,

$$f_{X_1|S}(x_1|s) = \frac{f_{X_1}(x_1)f_{S-1}(s-x_1)}{f_S(s)}$$

where $S_{-1} = S - X_1 \sim \text{Poisson}((n-1)\lambda)$. Therefore

$$f_{X_1|S}(0|s) = \frac{f_{X_1}(0)f_{S-1}(s)}{f_S(s)} = \frac{e^{-\lambda}(n-1)^s \lambda^s e^{-(n-1)\lambda}}{n^s \lambda^s e^{-n\lambda}} = \left(\frac{n-1}{n} \right)^s$$

and hence

$$U = \left(\frac{n-1}{n} \right)^S$$

To compute $E[U^k]$, we use the Poisson pgf: setting $t_n = (n-1)/n$, we have

$$E[U^k] = E \left[\left(t_n^k \right)^S \right] = \exp \left\{ n\lambda(t_n^k - 1) \right\}$$

so therefore

$$E[U] = \exp \left\{ n\lambda \left(\frac{n-1}{n} - 1 \right) \right\} = e^{-\lambda}$$

and

$$E[U^2] = \exp \left\{ n\lambda \left(\left(\frac{n-1}{n} \right)^2 - 1 \right) \right\} = \exp \left\{ \lambda \left(\frac{1-2n}{n} \right) \right\}$$

and therefore the variance is

$$\exp \left\{ \lambda \left(\frac{1-2n}{n} \right) \right\} - e^{-2\lambda} = e^{-2\lambda} (\exp\{\lambda/n\} - 1)$$

The ratio of variances is therefore

$$\frac{e^{-\lambda}(1 - e^{-\lambda})}{e^{-2\lambda}(\exp\{\lambda/n\} - 1)} = \frac{e^{\lambda} - 1}{\exp\{\lambda/n\} - 1} > 1$$

if $n > 1$.

(c) The variance of T is $4\text{Var}[X_1] = 4\theta^2/12 = \theta^2/3$. We also have that

$$U = E[T|S] = E[2X_1|X_{(n)}] = 2E[X_1|X_{(n)}] = X_{(n)}$$

as, given $X_{(n)}$, $X_1 \sim \text{Uniform}(0, X_{(n)})$. The distribution of $X_{(n)}$ is available from results for maximum order statistics

$$f_{X_{(n)}|\theta}(x|\theta) = \frac{nx^{n-1}}{\theta^n} \mathbb{1}_{(0,\theta)}(x)$$

with, therefore

$$E[U^k] = \frac{n}{\theta^n} \int_0^\theta x^{n+k-1} dx = \frac{n\theta^k}{n+k}$$

so that

$$E[U] = \frac{n}{n+1}\theta \quad E[U^2] = \frac{n\theta^2}{n+2} \quad \therefore \quad \text{Var}[U] = \frac{n\theta^2}{(n+1)^2(n+2)}$$

Therefore an unbiased estimator based on U is $(n+1)X_n/n$, which has variance

$$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3}$$

if $n > 1$.

8 The joint pdf is

$$f_{\underline{X}|\theta}(\underline{x}|\theta) = \frac{1}{\theta^n} \prod_{i=1}^n \mathbb{1}_{(\theta, 2\theta)}(X_i) = \frac{1}{\theta^n} \mathbb{1}_{(0, X_{(1)})}(\theta) \mathbb{1}_{(X_{(n)}/2, \infty)}(\theta)$$

so therefore $S = (X_{(1)}, X_{(n)})$ is a jointly sufficient statistic for θ . Also, by properties of the uniform distribution

$$E_{f_{T|\theta}}[T] = \frac{2}{3} \left(\theta + \frac{\theta}{2} \right) = \theta$$

and

$$\text{Var}_{f_{T|\theta}}[T] = \frac{4}{9} \frac{\theta^2}{12} = \frac{\theta^2}{27}$$

By the Rao-Blackwell theorem, we can improve the variance of the estimator by conditioning on the sufficient statistic. We have

$$U = E[T|S] = E[2X_1/3|X_{(1)}, X_{(n)}] = \frac{2}{3}E[X_1|X_{(1)}, X_{(n)}] = \frac{X_{(1)} + X_{(n)}}{3}$$

as, given $X_{(1)}, X_{(n)}$, $X_1 \sim \text{Uniform}(X_{(1)}, X_{(n)})$. The expectation of U is computed from the distribution of the order statistics. We have from the Q7 (c) and properties of expectations that

$$E[X_{(n)}] = \theta + \frac{n}{n+1}\theta = \frac{(2n+1)\theta}{n+1}$$

and from first principles, for $\theta < x < 2\theta$,

$$F_{X_{(1)}|\theta}(x|\theta) = 1 - \left(1 - \frac{x-\theta}{\theta}\right)^n = 1 - \left(\frac{2\theta-x}{\theta}\right)^n$$

so that

$$f_{X_{(1)}|\theta}(x|\theta) = n \frac{(2\theta-x)^{n-1}}{\theta^n} \mathbb{1}_{(\theta, 2\theta)}(x)$$

Therefore

$$\begin{aligned} E[X_{(1)}] &= \frac{n}{\theta^n} \int_{\theta}^{2\theta} x(2\theta-x)^{n-1} dx \\ &= \frac{1}{\theta^n} [-x(2\theta-x)^n]_{\theta}^{2\theta} + \frac{1}{\theta^n} \int_{\theta}^{2\theta} (2\theta-x)^n dx \\ &= \theta + \frac{1}{\theta^n} \left[-\frac{1}{n+1} (2\theta-x)^{n+1} \right]_{\theta}^{2\theta} \\ &= \theta + \frac{1}{(n+1)}\theta = \frac{(n+2)\theta}{n+1} \end{aligned}$$

Therefore U has expectation

$$\frac{1}{3} \left[\frac{(2n+1)\theta}{n+1} + \frac{(n+2)\theta}{n+1} \right] = \theta$$

and, by the Rao-Blackwell result, U has lower variance than T .

By standard methods, the joint density of $(U, V) = (X_{(1)}, X_{(n)})/\theta - 1$ is

$$f_{U,V}(u, v) = n(n-1)(v-u)^{n-2} \quad 0 < u < v < 1$$

Setting $R = (V - U)$ and $M = (U + V)$, so that $U = (M - R)/2$ and $V = (R + M)/2$, yielding Jacobian $1/2$, and joint pdf

$$f_{M,R}(m, r) = \frac{n(n-1)}{2} r^{n-2}$$

on the triangle defined by the inequalities

$$0 < (m - r) < (m + r) < 2.$$

The marginal for m is obtained by integrating out r from the joint. For $0 < m < 2$

$$\begin{aligned}
 f_M(m) &= \int_0^{\min\{m, 2-m\}} \frac{n(n-1)}{2} r^{n-2} dr = \frac{n(n-1)}{2} \left[\frac{r^{n-1}}{n-1} \right]_0^{\min\{m, 2-m\}} \\
 &= \frac{n}{2} (\min\{m, 2-m\})^{n-1} \\
 &= \begin{cases} \frac{n}{2} m^{n-1} & m \leq 1 \\ \frac{n}{2} (2-m)^{n-1} & 1 < m \leq 2 \end{cases}
 \end{aligned}$$

Therefore, by symmetry $E[M] = 1$, and

$$\begin{aligned}
 E[M^2] &= \int_0^1 \frac{n}{2} m^{n+1} dm + \int_1^2 \frac{n}{2} m^2 (2-m)^{n-1} dm \\
 &= \frac{n}{2(n+2)} + \frac{1}{2} \left[-m^2 (2-m)^n \right]_1^2 + \int_1^2 m (2-m)^n dm \\
 &= \frac{n}{2(n+2)} + \frac{1}{2} + \left[-\frac{1}{n+1} m (2-m)^{n+1} \right]_1^2 + \frac{1}{n+1} \int_1^2 (2-m)^{n+1} dm \\
 &= \frac{n}{2(n+2)} + \frac{1}{2} + \frac{1}{(n+1)} + \frac{1}{n+1} \left[-\frac{1}{n+2} (2-m)^{n+2} \right]_1^2 \\
 &= \frac{n}{2(n+2)} + \frac{1}{2} + \frac{1}{(n+1)} + \frac{1}{n+1} \frac{1}{n+2} \\
 &= \frac{n(n+1) + (n+1)(n+2) + 2(n+2) + 2}{2(n+1)(n+2)} = \frac{n^2 + 3n + 4}{(n+1)(n+2)}
 \end{aligned}$$

Hence the variance is

$$\text{Var}[M] = \frac{n^2 + 3n + 4 - (n+1)(n+2)}{(n+1)(n+2)} = \frac{2}{(n+1)(n+2)}$$

Now, the estimator of interest is

$$T = \frac{X_1 + X_n}{3} = \frac{\theta(U+V)}{3} + \theta = \frac{\theta M}{3} + \theta$$

so the variance of T is

$$\frac{2\theta^2}{9(n+1)(n+2)} < \frac{\theta^2}{27}$$

when $n > 1$.

9 (a) By invariance, the ML estimator of $\tau(\theta)$ is

$$\hat{\tau}(T) = \left(1 - \frac{T}{n} \right)^2 = 1 - \frac{2T}{n} + \frac{T^2}{n^2}$$

(b) We have that $T \sim \text{Binomial}(n, \theta)$, so $E[T] = n\theta$ and $E[T^2] = n\theta(1 - \theta) + n^2\theta^2$, so

$$E[\hat{\tau}(T)] = 1 - 2\theta + \frac{\theta(1 - \theta)}{n} + \theta^2 = (1 - \theta)^2 + \frac{\theta(1 - \theta)}{n}$$

so the bias is

$$\frac{\theta(1 - \theta)}{n}$$

(c) Again use Rao-Blackwell: the statistic

$$U = \mathbb{1}_{\{0\}}(X_1 + X_2)$$

is unbiased for τ , as

$$E_{f_U}[U] = \Pr[X_1 + X_2 = 0] = (1 - \theta)^2$$

Therefore if we can compute the expectation of U given T , then the resulting estimator will have lower variance than U .

$$E_{f_{U|T}}[U|T] = \Pr[X_1 + X_2 = 0|T]$$

Now,

$$\begin{aligned} \Pr[X_1 + X_2 = 0|T = t] &= \frac{\Pr\left[X_1 = 0, X_2 = 0, \sum_{i=3}^n X_i = t\right]}{\Pr\left[\sum_{i=1}^n X_i = t\right]} \\ &= \frac{(1 - \theta)^2 \binom{n-2}{t} \theta^t (1 - \theta)^{n-t-2}}{\binom{n}{t} \theta^t (1 - \theta)^{n-t}} \\ &= \frac{\binom{n-2}{t}}{\binom{n}{t}} = \frac{(n-2)!}{t!(n-2-t)!} \frac{t!(n-t)!}{n!} \\ &= \frac{(n-t)(n-t-1)}{n(n-1)} = \left(1 - \frac{t}{n}\right) \left(1 - \frac{t}{n-1}\right) \end{aligned}$$

Therefore the suggested estimator is

$$\left(1 - \frac{T}{n}\right) \left(1 - \frac{T}{n-1}\right) = 1 - \frac{T}{n} - \frac{T}{n-1} + \frac{T^2}{n(n-1)}$$

which has expectation

$$1 - \theta - \frac{n\theta}{n-1} + \frac{n\theta(1 - \theta) + n^2\theta^2}{n(n-1)} = (1 - \theta)^2$$

so this is the best unbiased estimator.