

MATH 557 - EXERCISES 2 SOLUTIONS

1 The likelihood in the original parameterization is

$$L(\theta_1, \theta_2 | x_1, x_2) = \binom{n_1}{x_1} \theta_1^{x_1} (1 - \theta_1)^{n_1 - x_1} \binom{n_2}{x_2} \theta_2^{x_2} (1 - \theta_2)^{n_2 - x_2}$$

If $\phi = \theta_2/(1 - \theta_2)$, then

$$\theta_2 = \phi/(1 + \phi) \quad \theta_1 = (\phi\psi)/(1 + \phi\psi).$$

Either by writing out the likelihood in full in the new parameterization, and maximizing in the usual way, or by using invariance properties of maximum likelihood estimates, we conclude that

$$\hat{\psi}(x_1, x_2) = \frac{\hat{\theta}_1/(1 - \hat{\theta}_1)}{\hat{\theta}_2/(1 - \hat{\theta}_2)} = \frac{(x_1/n_1)/(1 - x_1/n_1)}{(x_2/n_2)/(1 - x_2/n_2)} = \frac{x_1/(n_1 - x_1)}{x_2/(n_2 - x_2)}$$

2 $X_1, \dots, X_n \sim \text{Gamma}(\alpha, \beta)$ so that $E_{f_{X_i}}[X_i] = \alpha/\beta$ and $\text{Var}_{f_{X_i}}[X_i] = \alpha/\beta^2$ so that

$$E_{f_{X_i}}[X_i^2] = \text{Var}_{f_{X_i}}[X_i] + \{E_{f_{X_i}}[X_i]\}^2 = \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha(\alpha + 1)}{\beta^2}$$

Hence for the method of moments estimators $\hat{\alpha}_{MM}$ and $\hat{\beta}_{MM}$, need to solve the following:

$$\text{FIRST MOMENT} \quad \text{Solve} \quad \frac{1}{n} \sum_{i=1}^n x_i = \bar{x} = \frac{\alpha}{\beta}$$

$$\text{SECOND MOMENT} \quad \text{Solve} \quad \frac{1}{n} \sum_{i=1}^n x_i^2 = (\bar{x})^2 + S^2 = \frac{\alpha(\alpha + 1)}{\beta^2}$$

where $S^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$. Elementary algebra gives $\hat{\alpha}_{MM} = \frac{(\bar{x})^2}{S^2}$ and $\hat{\beta}_{MM} = \frac{\bar{x}}{S^2}$.

3 Writing $L(\theta)$ for $L(\theta|\underline{x})$ throughout:

(i) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^n x_i \right)^{\theta-1}$$

$$\text{STEP 2} \quad \log L(\theta) = n \log \theta + (\theta - 1) \sum_{i=1}^n \log x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta} + \sum_{i=1}^n \log x_i = 0 \quad \implies \quad \hat{\theta}_{ML} = -n / \sum_{i=1}^n \log x_i$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE: } \hat{\theta}_{ML} = -\frac{n}{\sum_{i=1}^n \log x_i} \quad \text{ESTIMATOR: } -\frac{n}{\sum_{i=1}^n \log X_i}$$

(ii) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n (\theta + 1) x_i^{-(\theta+2)} = (\theta + 1)^n \left(\prod_{i=1}^n x_i \right)^{-(\theta+2)}$$

$$\text{STEP 2} \quad \log L(\theta) = n \log(\theta + 1) - (\theta + 2) \sum_{i=1}^n \log x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta + 1} - \sum_{i=1}^n \log x_i = 0 \quad \implies \quad \hat{\theta}_{ML} = n / \sum_{i=1}^n \log x_i - 1$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{(\theta + 1)^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n \log x_i} - 1 \quad \text{ESTIMATOR: } = \frac{n}{\sum_{i=1}^n \log x_i} - 1$$

(iii) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n \theta^2 x_i \exp\{-\theta x_i\} = \theta^{2n} \left(\prod_{i=1}^n x_i \right) \exp\left\{-\theta \sum_{i=1}^n x_i\right\}$$

$$\text{STEP 2} \quad \log L(\theta) = 2n \log \theta + \sum_{i=1}^n \log x_i - \theta \sum_{i=1}^n x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{2n}{\theta} - \sum_{i=1}^n x_i = 0 \quad \implies \quad \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^n x_i}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{2n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{2n}{\sum_{i=1}^n x_i} \quad \text{ESTIMATOR: } = \frac{2n}{\sum_{i=1}^n X_i}$$

(iv) Because of the constraint in the pdf that $x \leq \theta$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i|\theta) = \begin{cases} \prod_{i=1}^n 2\theta^2 x_i^{-3} = 2^n \theta^{2n} \left(\prod_{i=1}^n x_i^{-3} \right) & \theta \leq x_1, \dots, x_n \\ 0 & \text{otherwise} \end{cases}$$

$$\text{STEP 2} \quad \log L(\theta) = n \log 2 + 2n \log \theta - 3 \sum_{i=1}^n \log x_i$$

At this point we note that the likelihood is monotonically increasing in θ , and hence the likelihood is maximized when θ is as **large** as possible but so that the constraint $\theta \leq x_1, \dots, x_n$ is still satisfied, hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \min\{x_1, \dots, x_n\} \quad \text{ESTIMATOR: } \min\{X_1, \dots, X_n\}$$

(v) For $\theta > 0$

$$\text{STEP 1} \quad L(\theta) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n \left(\frac{\theta}{2}\right) \exp\{-\theta|x_i|\} = 2^{-n}\theta^n \exp\left\{-\theta \sum_{i=1}^n |x_i|\right\}$$

$$\text{STEP 2} \quad \log L(\theta) = -n \log 2 + n \log \theta - \theta \sum_{i=1}^n |x_i|$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\theta)\} = \frac{n}{\theta} - \sum_{i=1}^n |x_i| = 0 \quad \Rightarrow \quad \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n |x_i|}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\theta)\} = -\frac{n}{\theta^2} < 0 \quad \text{for all } \theta$$

Hence

$$\text{ESTIMATE : } \hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^n |x_i|} \quad \text{ESTIMATOR : } \frac{n}{\sum_{i=1}^n |x_i|}$$

(vi) Because of the constraint in the pdf that $\theta_1 \leq x \leq \theta_2$

$$\text{STEP 1} \quad L(\theta_1, \theta_2) = \prod_{i=1}^n f_X(x_i|\theta) = \begin{cases} \prod_{i=1}^n \frac{1}{(\theta_2 - \theta_1)} = \frac{1}{(\theta_2 - \theta_1)^n} & \theta_1 \leq x_1, \dots, x_n \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{STEP 2} \quad \log L(\theta_1, \theta_2) = -n \log (\theta_2 - \theta_1)$$

At this point we note that the likelihood is monotonically increasing in θ_1 and monotonically decreasing in θ_2 , and hence the likelihood is maximized when θ_1 is as **large** as possible and when θ_2 is as **small** as possible, but so that the constraint $\theta_1 \leq x_1, \dots, x_n \leq \theta_2$ (and $\theta_2 \geq \theta_1$) is still satisfied, hence

$$\begin{array}{ll} \text{ESTIMATE : } \hat{\theta}_{1ML} = \min \{x_1, \dots, x_n\} & \theta_1 = \min \{X_1, \dots, X_n\} \\ \hat{\theta}_{2ML} = \max \{x_1, \dots, x_n\} & \theta_2 = \max \{X_1, \dots, X_n\} \end{array}$$

(vii) Noting the constraint in the pdf that $x \geq \theta_2$, we have

$$\text{STEP 1} \quad L(\theta_1, \theta_2) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n \theta_1 \theta_2^{\theta_1} x_i^{-(\theta_1+1)} = \theta_1^n \theta_2^{n\theta_1} \left(\prod_{i=1}^n x_i\right)^{-(\theta_1+1)}$$

$$0 \leq x_1, \dots, x_n \leq \theta_2$$

$$\text{STEP 2} \quad \log L(\theta_1, \theta_2) = n \log \theta_1 + n\theta_1 \log \theta_2 - (\theta_1 + 1) \sum_{i=1}^n \log x_i$$

$$\text{STEP 3} \quad \frac{\partial}{\partial \theta_1} \{\log L(\theta_1, \theta_2)\} = \frac{n}{\theta_1} + n \log \theta_2 - \sum_{i=1}^n \log x_i = 0$$

$$\Rightarrow \quad \hat{\theta}_{1ML} = \frac{n}{\sum_{i=1}^n \log x_i - n \log \hat{\theta}_{2ML}}$$

$$\frac{\partial}{\partial \theta_2} \{\log L(\theta_1, \theta_2)\} = \frac{n\theta_1}{\theta_2}$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when θ_2 is as **large** as possible, that is, when $\hat{\theta}_{2ML} = \min \{x_1, \dots, x_n\}$. Hence

$$\begin{array}{ll} \text{ESTIMATES} & \hat{\theta}_{1ML} = \frac{n}{\left[\sum_{i=1}^n \log x_i - n \log \{ \min \{x_1, \dots, x_n\} \} \right]} \\ & \hat{\theta}_{2ML} = \min \{x_1, \dots, x_n\} \\ \text{ESTIMATORS:} & \theta_1 \quad \frac{n}{\left[\sum_{i=1}^n \log X_i - n \log \{ \min \{X_1, \dots, X_n\} \} \right]} \\ & \theta_2 \quad \min \{X_1, \dots, X_n\} \end{array}$$

4 For the *Poisson*(λ) case

$$\text{STEP 1} \quad L(\lambda) = \prod_{i=1}^n f_X(x_i|\lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\left(\prod_{i=1}^n x_i! \right)}$$

$$\text{STEP 2} \quad \log L(\lambda) = - \sum_{i=1}^n \log x_i! - n\lambda + \left(\sum_{i=1}^n x_i \right) \log \lambda$$

$$\text{STEP 3} \quad \frac{d}{d\lambda} \{ \log L(\lambda) \} = -n + \left(\sum_{i=1}^n x_i \right) \frac{1}{\lambda} = 0 \quad \implies \quad \hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$\text{STEP 4} \quad \frac{d^2}{d\lambda^2} \{ \log L(\lambda) \} = - \left(\sum_{i=1}^n x_i \right) \frac{1}{\lambda^2} < 0 \quad \text{for all } \lambda$$

Therefore

$$\text{ESTIMATE : } \hat{\lambda}_{ML} = \bar{x} \quad \text{ESTIMATOR: } \bar{X}$$

We see that

$$T_1 = \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \implies \quad E_{f_{T_1}} [T_1] = \lambda$$

and T_1 is unbiased. Now, if

$$\tau = \tau(\lambda) = e^{-\lambda} \quad \text{so that} \quad \lambda = -\log \tau$$

and we can reformulate the likelihood in terms of τ , giving

$$\log L(\tau) = - \sum_{i=1}^n \log x_i! + n \log \tau + \left(\sum_{i=1}^n x_i \right) \log(-\log \tau)$$

Differentiating wrt τ in the usual way and equating to zero, we find that $\hat{\tau}_{ML} = \exp\{-\bar{x}\}$, so that $\hat{\tau}_{ML}(\lambda) = \tau(\hat{\lambda}_{ML})$. Using a Taylor approximation

$$g(\hat{\lambda}) \approx g(\lambda) + \dot{g}(\lambda)(\hat{\lambda} - \lambda) + \ddot{g}(\lambda)(\hat{\lambda} - \lambda)^2/2$$

with $g(t) = e^{-t}$ and $\hat{\lambda} = \bar{x}$, and taking expectations, noting that $E_{f_X}[\bar{X}] = \lambda$,

$$E_{f_X}[\exp\{-\bar{X}\}] \approx e^{-\lambda} + e^{-\lambda} \text{Var}[\bar{X}]/n = e^{-\lambda} + \frac{1}{2}e^{-\lambda}\lambda/n$$

so the bias is approximately $\frac{1}{2}e^{-\lambda}\lambda/n$.

5 Noting the constraint in the pdf that $x \geq \eta$, we have

$$\text{STEP 1} \quad L(\lambda, \eta) = \prod_{i=1}^n f_X(x_i|\theta) = \prod_{i=1}^n \lambda \exp\{-\lambda(x_i - \eta)\} = \lambda^n \exp\left\{-\lambda \sum_{i=1}^n (x_i - \eta)\right\} \\ \eta \leq x_1, \dots, x_n, \text{ zero otherwise}$$

$$\text{STEP 2} \quad \log L(\lambda, \eta) = n \log \lambda - \lambda \sum_{i=1}^n (x_i - \eta) = n \log \lambda - \lambda \sum_{i=1}^n x_i + n\lambda\eta$$

$$\text{STEP 3} \quad \frac{\partial}{\partial \lambda} \{\log L(\lambda, \eta)\} = \frac{n}{\lambda} + \sum_{i=1}^n x_i - n\eta = 0 \quad \implies \quad \hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i - n\hat{\eta}}$$

$$\frac{\partial}{\partial \eta} \{\log L(\lambda, \eta)\} = n\lambda$$

The second of the partial derivative equations indicates again that the maximum of the likelihood occurs when η is as **large** as possible, that is, when $\hat{\eta} = \min\{x_1, \dots, x_n\}$. Hence

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^n x_i - n \min\{x_1, \dots, x_n\}} = \frac{n}{\sum_{i=1}^n (x_i - \min\{x_1, \dots, x_n\})}$$

ESTIMATES

$$\hat{\eta}_{ML} = \min\{x_1, \dots, x_n\}$$

with the obvious corresponding estimators.

6 $X_1, \dots, X_n \sim \text{Exponential}(1/\theta)$ so that $E_{f_{X_i}}[X_i] = \theta$ and hence, using standard mgf techniques, we have

$$X = \sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\theta) \quad \implies \quad E_{f_X}[X] = n/(1/\theta) = n\theta$$

so that if $T_1 = \bar{X} = \frac{1}{n}X$ then

$$E_{f_{T_1}}[T_1] = \frac{1}{n}n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ .

Now if $Y_1 = \min\{X_1, \dots, X_n\}$, then previous order statistics results give that

$$F_{Y_1}(y) = 1 - \{1 - F_X(y)\}^n = 1 - \left\{1 - \left(1 - e^{-y/\theta}\right)\right\}^n = 1 - e^{-ny/\theta} \quad y > 0$$

so that $Y_1 \sim \text{Exponential}(n/\theta)$. Hence if $Z = nY_1$ then

$$E_{f_{Y_1}}[Y_1] = \frac{\theta}{n} \quad \implies \quad E_{f_Z}[Z] = n \frac{\theta}{n} = \theta$$

and hence Z is an unbiased estimator of θ .

7 We have

$$f_X(x) = \frac{1}{2} \quad \theta - 1 \leq x \leq \theta + 1 \quad F_X(x) = \frac{x - (\theta - 1)}{2} = \frac{x - \theta + 1}{2} \quad \theta - 1 \leq x \leq \theta + 1$$

$E_{f_{X_i}}[X_i] = \theta$ (by integration, or by noting that the pdf is constant and hence symmetric about θ) and hence, using standard expectation techniques, we have that if $T_1 = \bar{X}$

$$E_{f_{T_1}}[T_1] = \frac{1}{n} \sum_{i=1}^n E_{f_{X_i}}[X_i] = \frac{1}{n} \sum_{i=1}^n \theta = \frac{1}{n} n\theta = \theta$$

and hence T_1 is an unbiased estimator of θ .

Now if $Y_1 = \min\{X_1, \dots, X_n\}$ and $Y_n = \max\{X_1, \dots, X_n\}$, then previous order statistics results give that

$$f_{Y_1}(y) = n f_X(y) \{1 - F_X(y)\}^{n-1} = n \frac{1}{2} \left\{1 - \frac{y - (\theta - 1)}{2}\right\}^{n-1} = \frac{n}{2} \left\{\frac{1 + \theta - y}{2}\right\}^{n-1} \quad \theta - 1 \leq y \leq \theta + 1$$

and

$$f_{Y_n}(y) = n f_X(y) \{F_X(y)\}^{n-1} = n \frac{1}{2} \left\{\frac{y - (\theta - 1)}{2}\right\}^{n-1} = \frac{n}{2} \left\{\frac{1 - \theta + y}{2}\right\}^{n-1} \quad \theta - 1 \leq y \leq \theta + 1$$

For the expectations,

$$\begin{aligned} E_{f_{Y_1}}[Y_1] &= \int_{\theta-1}^{\theta+1} y \frac{n}{2} \left\{\frac{1 + \theta - y}{2}\right\}^{n-1} dy \\ &= \frac{n}{2} \int_0^1 ((1 + \theta) - 2t) t^{n-1} 2 dt \quad \text{setting } t = (1 + \theta - y)/2 \implies y = (1 + \theta) - 2t \\ &= (1 + \theta) \int_0^1 n t^{n-1} dt - 2n \int_0^1 t^n dt = (1 + \theta) - \frac{2n}{n+1} \end{aligned}$$

and

$$\begin{aligned} E_{f_{Y_n}}[Y_n] &= \int_{\theta-1}^{\theta+1} y \frac{n}{2} \left\{\frac{1 - \theta + y}{2}\right\}^{n-1} dy \\ &= \frac{n}{2} \int_0^1 (2t - (1 - \theta)) t^{n-1} 2 dt \quad \text{setting } t = (1 - \theta + y)/2 \implies y = 2t - (1 - \theta) \\ &= 2 \int_0^1 n t^n dt - (1 - \theta) \int_0^1 n t^{n-1} dt = \frac{2n}{n+1} - (1 - \theta) \end{aligned}$$

so that if $M = (Y_1 + Y_n)/2$ then by properties of expectations

$$E_{f_M}[M] = \frac{1}{2} E_{f_{Y_1}}[Y_1] + \frac{1}{2} E_{f_{Y_n}}[Y_n] = \left[\frac{1}{2} (1 + \theta) - \frac{n}{n+1} \right] + \left[\frac{n}{n+1} - \frac{1}{2} (1 - \theta) \right] = \theta$$

and hence M is an unbiased estimator for θ .

8 (i) For $\lambda > 0$

$$\text{STEP 1} \quad L(\lambda) = \prod_{i=1}^n f_X(x_i|\lambda) = \prod_{i=1}^n \frac{\lambda^2}{\Gamma(2)} x_i \exp\{-\lambda x_i\} = \frac{\lambda^{2n}}{\{\Gamma(2)\}^n} \left(\prod_{i=1}^n x_i\right) \exp\left\{-\lambda \sum_{i=1}^n x_i\right\}$$

$$\text{STEP 2} \quad \log L(\lambda) = 2n \log \lambda - n \log \Gamma(2) + \sum_{i=1}^n \log x_i - \lambda \sum_{i=1}^n x_i$$

$$\text{STEP 3} \quad \frac{d}{d\theta} \{\log L(\lambda)\} = \frac{2n}{\lambda} - \sum_{i=1}^n x_i = 0 \quad \implies \quad \hat{\lambda}_{ML} = \frac{2n}{\sum_{i=1}^n x_i}$$

$$\text{STEP 4} \quad \frac{d^2}{d\theta^2} \{\log L(\lambda)\} = -\frac{2n}{\lambda^2} < 0 \quad \text{at } \lambda = \hat{\lambda}_{ML}$$

Hence the estimator is $2n / \left(\sum_{i=1}^n X_i\right)$

(ii) By the invariance property of maximum likelihood estimators, we must have that the ML estimator of $\tau = 1/\lambda$ is $T = \sum_{i=1}^n X_i / 2n$. Now, using mgfs, it is straightforward to show that

$$\sum_{i=1}^n X_i \sim \text{Gamma}(2n, \lambda) \quad \implies \quad T = \frac{1}{2n} \sum_{i=1}^n X_i \sim \text{Gamma}(2n, 2n\lambda)$$

so that

$$E_{f_T}[T] = \frac{2n}{2n\lambda} = \frac{1}{\lambda} \quad \text{Var}_{f_T}[T] = \frac{2n}{(2n\lambda)^2} = \frac{1}{2n\lambda^2} \quad E_{f_T}[T^2] = \frac{1}{2n\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \frac{2n+1}{2n\lambda^2}$$

9 (a) We have for the likelihood

$$L(\theta|\underline{x}) = \prod_{i=1}^n \frac{1}{\pi} \frac{1}{1 + (x_i - \theta)^2} = \frac{1}{\pi^n} \prod_{i=1}^n \frac{1}{1 + (x_i - \theta)^2}$$

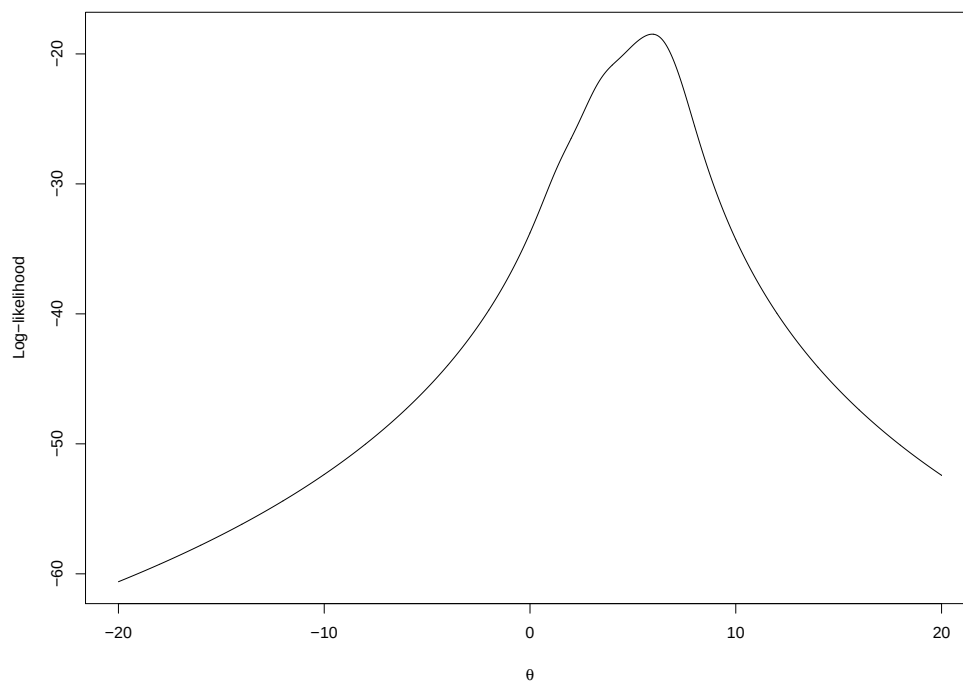
$$l(\theta|\underline{x}) = -n \log \pi - \sum_{i=1}^n \log(1 + (x_i - \theta)^2).$$

yielding the following score equation (which cannot be solved analytically)

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^n \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} = 0$$

(b) The code below evaluates the log-likelihood in R on a fine grid on the range (-20,20):

```
x.dat<-c(7.36,5.14,3.71,3.15,6.00,6.38,1.34,6.73);thvec<-seq(-20,20,length=20001)
log.pdf.func<-function(x,th){return(-log(pi)-log(1+(x-th)^2))}
log.pdf.mat<-outer(thvec,x.dat,log.pdf.func)
log.like.vec<-apply(log.pdf.mat,1,sum)
plot(thvec,log.like.vec,type='l',xlab=expression(theta),ylab='Log-likelihood')
th.max<-thvec[which.max(log.like.vec)]
```



In this code, the variable `th.max` contains the numerical value of the estimate of θ ; to two decimal places, we find that $\hat{\theta}(x) = 5.94$. The code also produces the following plot. The following code uses the `optimize` function to produce the same result: the commands

```
log.like.fx<-function(x,xd){return(sum(log.pdf.func(xd,x[1])))}
optimize(f=log.like.fx,xd=x.dat,interval=c(-20,20),maximum=TRUE)

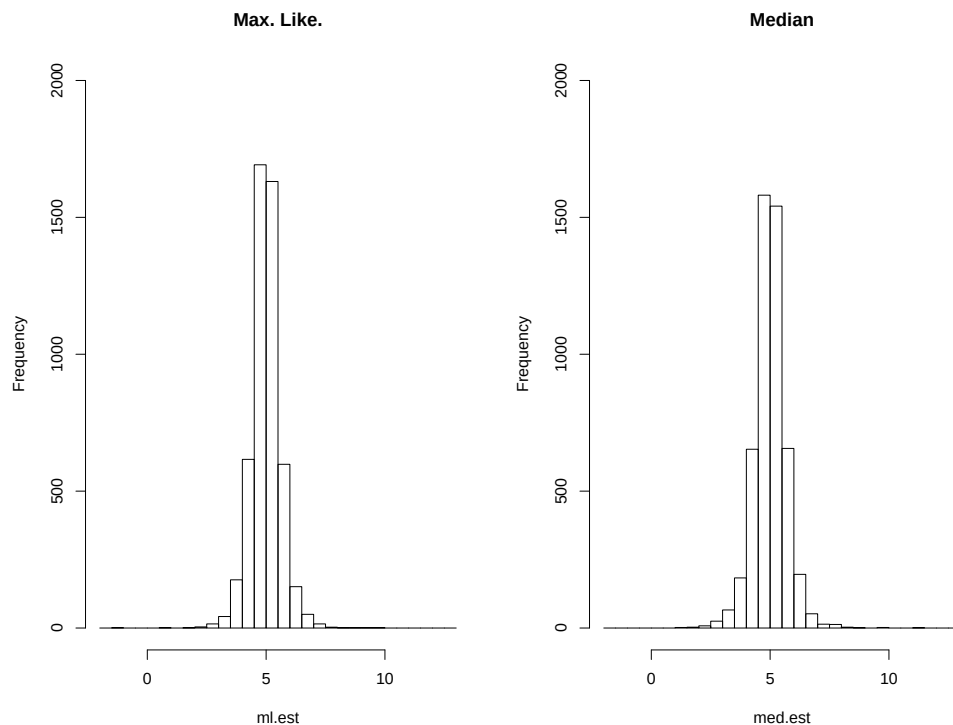
yield

$maximum [1] 5.93949
$objective [1] -18.48059
```

10 The following R code computes the $N = 5000$ estimates.

```
N<-5000
n<-8
ml.est<-med.est<-rep(0,N)
theta0<-5
for(i in 1:N){
  x<-rcauchy(n)+theta0
  ml<-optimize(f=log.like.fx,xd=x,interval=c(-20,20),maximum=TRUE)
  ml.est[i]<-ml$maximum
  med.est[i]<-median(x)
}
```

The histograms below depict the 5000 maximum likelihood and median estimates:



It can be discerned from these figures that the variance of the median estimator is slightly larger than for the maximum likelihood estimator; in one simulation, I got that the variance for the sample of median estimates was 0.468, and the variance for the sample of maximum likelihood estimates was 0.387.