

MATH 557 - EXERCISES 1 : SOLUTIONS

- 1 We construct the joint pmf/pdf in each case, and inspect the required conditional pdf. Note that 1-1 transformations of the statistics are also sufficient.

(a) For the

$$f_{\underline{X}|\alpha,\beta}(\underline{x}|\alpha,\beta) = \left\{ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \right\}^n \left\{ \prod_{i=1}^n x_i \right\}^{\alpha-1} \left\{ \prod_{i=1}^n (1-x_i) \right\}^{\beta-1}$$

suggesting the sufficient statistic

$$\underline{T}(\underline{X}) = \left(\prod_{i=1}^n x_i, \prod_{i=1}^n (1-x_i) \right)$$

and the result follows using the Neyman Factorization Theorem.

- (b) Writing $\lambda = \log \theta$, we realize that this is the *Poisson*($\log \theta$) model. Hence by properties of the Exponential Family

$$T(\underline{X}) = \sum_{i=1}^n X_i$$

is a sufficient statistic for $\log \theta$.

(c) We have

$$\begin{aligned} f_{\underline{X}|\theta}(\underline{x}|\theta) &= \frac{1}{\theta} \quad \theta < x_1, \dots, x_n < 2\theta \\ &= \frac{1}{\theta} I_{(x_{(n)}/2, x_{(1)})}(\theta) \end{aligned}$$

suggesting the sufficient statistic

$$\underline{T}(\underline{X}) = (X_{(1)}, X_{(n)})$$

and the result follows using the Neyman Factorization Theorem.

- 2 (a) We have

$$f_{\underline{X}|\lambda}(\underline{x}|\lambda) = \lambda^n \exp \left\{ -\lambda \sum_{i=1}^n x_i \right\} = \lambda^n \exp \{ -\lambda T(\underline{x}) \}$$

say, so that for two points $\underline{x}$ and $\underline{y}$ the ratio

$$\frac{f_{\underline{X}|\lambda}(\underline{x}|\lambda)}{f_{\underline{X}|\lambda}(\underline{y}|\lambda)} = \exp \left\{ -\lambda (T(\underline{x}) - T(\underline{y})) \right\}$$

which is a constant if and only if $T(\underline{x}) = T(\underline{y})$. Therefore $T(\underline{x})$ is minimal sufficient.

- (b) Given $x_{(1)}, \dots, x_{(m)}$, we can construct the joint pdf for the order statistic data by noting that if $X_{(m)} = x_{(m)}$, then we have $X_{(r)} > x_{(m)}$ for the $n - m$ order statistics $X_{(r)}$, $r = m + 1, \dots, n$. Thus, as the “survivor” function takes the form $1 - F_{X|\lambda}(x|\lambda) = e^{-\lambda x}$, we have

$$f_{\underline{X}|\lambda}(\underline{x}|\lambda) = m! \binom{n}{m} \times \lambda^m \exp \left\{ -\lambda \sum_{i=1}^m x_{(i)} \right\} \times \exp \{ -(n - m)\lambda x_{(m)} \}$$

where the combinatorial term counts the number of possible arrangements of the random sample points. Thus a sufficient statistic is

$$T(\underline{X}) = \sum_{i=1}^m X_{(i)} + (n - m)X_{(m)}$$

by the Neyman Factorization Theorem.

3 We have for $t = 0, 1, \dots$,

$$f_{\underline{X}|\theta}(\underline{x}|\theta) = \frac{\theta^{\sum_{i=1}^n x_i} e^{-n\theta}}{\prod_{i=1}^n x_i!}$$

and $T(\underline{X}) \sim \text{Poisson}(n\theta)$ from distributional results, so that

$$f_{\underline{X}|T(\underline{X})}(\underline{x}|t) = \frac{\theta^{\sum_{i=1}^n x_i} e^{-n\theta} / \prod_{i=1}^n x_i!}{(n\theta)^t e^{-n\theta} / t!} = \frac{t!}{x_1! \dots x_n!} \left(\frac{1}{n} \right)^t \quad \underline{x} \in A_t$$

and zero otherwise, where

$$A_t \equiv \{ \underline{x} : x_1 + \dots + x_n = t \}.$$

- 4 (a) Suppose, for two points $\underline{x}$ and $\underline{y}$ in the parameter space, $c(\underline{x}, \underline{y})$ is a function of these two arguments. We have that

$$\begin{aligned} \frac{f_i(\underline{x})}{f_i(\underline{y})} = c(\underline{x}, \underline{y}), \quad i = 0, \dots, k &\iff \frac{f_i(\underline{x})}{f_i(\underline{y})} = \frac{f_0(\underline{x})}{f_0(\underline{y})}, \quad i = 1, \dots, k \\ &\iff \frac{f_i(\underline{x})}{f_0(\underline{x})} = \frac{f_i(\underline{y})}{f_0(\underline{y})}, \quad i = 1, \dots, k \\ &\iff T_i(\underline{x}) = T_i(\underline{y}) \end{aligned}$$

for the given statistic

$$T(\underline{X}) = (T_1(\underline{X}), \dots, T_k(\underline{X})) = (T_i(\underline{X}))_{i=1, \dots, k}$$

Therefore, by the Theorem from lectures, $T(\underline{X})$ is minimal sufficient. This result implies that for selecting a *model* from a group of models, likelihood ratios provide summary statistics.

- (b) If $\underline{T}'(\underline{X})$ is sufficient for θ , then by the Neyman Factorization Theorem there exist g and h such that

$$f_{\theta}(\underline{x}) = g(\underline{T}'(\underline{x}); \theta) h(\underline{x}).$$

Thus, choosing $\theta_i, i = 1, \dots, k$, such that $f_i = f_{\theta_i}$,

$$\underline{T}(\underline{X}) = \left(\frac{f_i(\underline{X})}{f_0(\underline{X})} \right)_{i=1, \dots, k} = \left(\frac{g_i(\underline{T}'(\underline{X}); \theta_i) h(\underline{X})}{g_0(\underline{T}'(\underline{X}); \theta_0) h(\underline{X})} \right)_{i=1, \dots, k} = \left(\frac{g_i(\underline{T}'(\underline{X}); \theta_i)}{g_0(\underline{T}'(\underline{X}); \theta_0)} \right)_{i=1, \dots, k}$$

Hence for any sufficient $\underline{T}'$, $\underline{T}$ is a function of $\underline{T}'$. Therefore $\underline{T}$ is minimal sufficient.

- (c) Let T be minimal sufficient for θ , so that if W is also sufficient for θ , there exists h such that $h(W) = T$. Let $T' = r(T)$ where r is a 1-1 mapping. Then $T' = r(h(W))$, so T' is a function of every sufficient statistic.
- (d) In the notation of the earlier parts, let $\mathcal{F}_1 = \{f_{\beta} : \beta = 1, 2\}$ and $\mathcal{F} = \{f_{\beta} : \beta > 0\}$, where f_{β} is the Exponential density with expectation β , so that

$$f_{\beta}(x) = \frac{1}{\beta} e^{-x/\beta} I_{(0, \infty)}(x).$$

If $X_{(1)} = \min\{X_1, \dots, X_n\}$, define

$$T(\underline{X}) = \frac{f_2(\underline{X})}{f_1(\underline{X})} = \frac{\frac{1}{2^n} e^{-\frac{1}{2} \sum_{i=1}^n X_i} I_{(0, \infty)}(X_{(1)})}{\frac{1}{e^{-\sum_{i=1}^n X_i} I_{(0, \infty)}(X_{(1)})}} = 2^{-n} \exp \left\{ \frac{n\bar{X}}{2} \right\}.$$

However, since

$$f_{\beta}(\underline{x}) = \frac{1}{\beta^n} e^{-n\bar{x}/\beta} I_{(0, \infty)}(x_{(1)}) = \frac{1}{(2\beta)^n} \frac{1}{T(\underline{x})^{2/\beta}} I_{(0, \infty)}(x_{(1)})$$

$T(\underline{X})$ is sufficient for β by the Neyman factorization theorem. Therefore $T(\underline{X})$ is minimal sufficient for β by part (b). Since $T(\cdot)$ is a 1-1 transformation, $\bar{X}$ is minimal sufficient for β by part (c).

- 5 (a) The joint pmf takes the form of an n -sample multinomial without the combinatorial term (here we observe the X_i s individually, not merely the totals in each of the categories):

$$f_{\underline{X}|\underline{\theta}}(\underline{x}|\underline{\theta}) = \prod_{j=1}^3 \theta_j^{\sum_{i=1}^n I_{\{j\}}(x_i)} = \prod_{j=1}^3 \theta_j^{n_j}$$

where

$$n_j = \sum_{i=1}^n I_{\{j\}}(x_i)$$

counts the number of times X_i takes the value j , for $j = 1, 2, 3$. But $n_3 = n - n_1 - n_2$, hence

$$f_{\underline{X}|\underline{\theta}}(\underline{x}|\underline{\theta}) = \theta_1^{n_1} \theta_2^{n_2} (1 - \theta_1 - \theta_2)^{n - n_1 - n_2}$$

Thus, by the Neyman Factorization Theorem, $\underline{T} = (N_1, N_2)$ is sufficient for $\underline{\theta} = (\theta_1, \theta_2)$.

(b) When looking at N_1 and N_2 , we have the traditional multinomial pmf, so

$$f_{\underline{T}(\underline{X})|\underline{\theta}}(\underline{T}(\underline{x})|\underline{\theta}) = \frac{n!}{n_1!n_2!(n - n_1 - n_2)!} \theta_1^{n_1} \theta_2^{n_2} (1 - \theta_1 - \theta_2)^{n_3}$$

where $\underline{T}(\underline{x}) = (n_1, n_2)$ and $n_3 = n - n_1 - n_2$.

(c) Using the using calculus approach, we form the log likelihood, partially differentiate in turn with respect to θ_1 and θ_2 , equate to zero, and then solve the resulting two equations simultaneously. We have

$$l(\theta_1, \theta_2|\underline{x}) = c(\underline{n}) + n_1 \log \theta_1 + n_2 \log \theta_2 + n_3 \log(1 - \theta_1 - \theta_2)$$

so that

$$\begin{aligned} \frac{\partial l}{\partial \theta_1} &= \frac{n_1}{\theta_1} - \frac{n_3}{1 - \theta_1 - \theta_2} \\ \frac{\partial l}{\partial \theta_2} &= \frac{n_2}{\theta_2} - \frac{n_3}{1 - \theta_1 - \theta_2} \end{aligned}$$

Equating to zero and subtracting the second from the first equation, we obtain that

$$\frac{n_1}{\theta_1} = \frac{n_2}{\theta_2} \quad \therefore \quad \frac{\theta_1}{\theta_2} = \frac{n_1}{n_2}.$$

Substituting back into the first equation, we have

$$\frac{n_1}{n_3} = \frac{\theta_1}{1 - \theta_1 - \theta_2} = \frac{\theta_1}{1 - \theta_1 - n_2 \theta_1 / n_1}$$

Cross multiplying, we get

$$n_1 - n_1 \theta_1 - n_2 \theta_1 = n_3 \theta_1$$

and hence

$$n_1 = (n_1 + n_2 + n_3) \theta_1 \quad \therefore \quad \theta_1 = \frac{n_1}{n_1 + n_2 + n_3} = \frac{n_1}{n}$$

Thus, by a similar argument for θ_2 , we deduce

$$\hat{\underline{\theta}}(\underline{x}) = (n_1/n, n_2/n).$$

and hence the estimator is

$$\hat{\underline{\theta}}(\underline{X}) = (N_1/n, N_2/n).$$

Note that the case covered here presumes $n_1, n_2, n_3 > 0$. If this does not hold, the maximum likelihood estimator does not lie in the parameter space and is not defined.