

MATH 557 - EXERCISES 3

These exercises are not for assessment

1 Suppose that $X \sim \text{Binomial}(n, \theta)$ for $0 < \theta < 1$.

- (a) Verify that the estimator $T(X) = X/n$ is unbiased for θ .
- (b) Consider $\tau(\theta) = 1/\theta$. Find an unbiased estimator of $\tau(\theta)$.

2 Suppose that $X_1, \dots, X_n \sim \text{Uniform}(0, \theta)$ for $\theta > 0$ is a random sample.

- (a) Find an unbiased estimator, $T(\underline{X})$, of θ .
- (b) Compute the variance/mean square error of $T(\underline{X})$.
- (c) Compute the amount by which the variance of $T(\underline{X})$ exceeds the Cramér-Rao lower bound for this non-regular model.

3 Suppose that $X_1, \dots, X_n \sim \text{Beta}(\theta, 1)$ for $\theta > 0$ is a random sample, so that

$$f_{X|\theta}(x|\theta) = \theta x^{\theta-1} \quad 0 < x < 1$$

and zero otherwise.

- (a) Find the maximum likelihood estimator of θ , to be denoted $\hat{\theta}_n(\underline{X})$.
- (b) Is $\hat{\theta}_n(\underline{X})$ unbiased for θ ? Justify your answer.
- (c) Does $\hat{\theta}_n(\underline{X})$ attain the Cramér-Rao lower bound? Justify your answer.

4 Suppose that $X_1, \dots, X_n \sim \text{Normal}(0, \theta)$. Find the best unbiased estimator of θ .

5 Suppose that $f_{X|\theta}(x) = f(x - \theta)$ is a location family distribution.

- (a) Show that for this class of models, the Fisher information for θ , $\mathcal{I}(\theta)$, does not depend on θ .
- (b) Compute $\mathcal{I}(\theta)$ if

$$f(x) = \frac{1}{2}e^{-|x|} \quad -\infty < x < \infty$$

- (c) Compute $\mathcal{I}(\theta)$ if

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2} \quad -\infty < x < \infty$$

6 Suppose that $X_1, \dots, X_n$ is a random sample from the distribution with pdf

$$f_{X|\theta}(x|\theta) = \frac{3\theta^3}{(x + \theta)^4} \quad 0 < x < \infty$$

and zero otherwise, for parameter $\theta > 0$.

Find an unbiased estimator for θ , and the variance of this estimator.

7 Using the Rao-Blackwell Theorem in the following examples, demonstrate the construction of an estimator U defined by $U = E[T|S] = g(S)$, where T is an unbiased estimator of $\tau(\theta)$, and S is sufficient for θ , which is unbiased and has variance not greater than T .

(a) If $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$, $\tau(\lambda) = \lambda$

$$S = \sum_{i=1}^n X_i \quad T = X_1$$

(b) If $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$, $\tau(\lambda) = e^{-\lambda}$

$$S = \sum_{i=1}^n X_i \quad T = \mathbb{1}_{\{0\}}(X_1)$$

(c) If $X_1, \dots, X_n \sim \text{Uniform}(0, \theta)$, $\tau(\theta) = \theta$

$$S = \max\{X_1, \dots, X_n\} \quad T = 2X_1$$

Demonstrate the improvement in variance using simulation if necessary.

8 Suppose $X_1, \dots, X_n$ are independent random variables uniformly distributed over $(\theta, 2\theta)$. Find a (minimal) sufficient statistic S , and that $T = 2X_1/3$ is an unbiased estimator of θ . Using the Rao-Blackwell Theorem, find an unbiased estimator of θ with variance no greater than T .

9 Suppose $X_1, \dots, X_n$ are independent random variables uniformly distributed as $\text{Bernoulli}(\theta)$. Let $\tau(\theta) = (1 - \theta)^2$.

(a) Find the ML estimator of $\tau(\theta)$ as a function of the τ -sufficient statistic $T = \sum_{i=1}^n X_i$.

(b) Show that T is biased for τ .

(c) Find the best unbiased estimator of τ , recalling that T is a complete statistic in this family.