

MATH 557 - EXERCISES 2

These exercises are not for assessment

- 1 Suppose that $X_1 \sim \text{Binomial}(n_1, \theta_1)$ and $X_2 \sim \text{Binomial}(n_2, \theta_2)$ be independent random variables. Derive the maximum likelihood estimator of the odds ratio ψ defined by

$$\psi = \frac{\theta_1/(1 - \theta_1)}{\theta_2/(1 - \theta_2)}$$

Hint : write down the likelihood in terms of the 1-1 reparameterization

$$(\theta_1, \theta_2) \longrightarrow (\phi, \psi)$$

for some appropriately chosen parameter ϕ .

- 2 Suppose that $X_1, \dots, X_n$ are a random sample from a $\text{Gamma}(\alpha, \beta)$ distribution. Find the method of moments estimators of α and β .

- 3 Find the maximum likelihood estimators of the unknown parameters in the following probability densities on the basis of a random sample of size n .

(i) $f_{X|\theta}(x|\theta) = \theta x^{\theta-1}$, $0 < x < 1, \theta > 0$.

(ii) $f_{X|\theta}(x|\theta) = (\theta + 1)x^{-\theta-2}$, $1 < x, \theta > 0$.

(iii) $f_{X|\theta}(x|\theta) = \theta^2 x \exp\{-\theta x\}$, $0 < x, \theta > 0$.

(iv) $f_{X|\theta}(x|\theta) = 2\theta^2 x^{-3}$, $\theta \leq x, \theta > 0$.

(v) $f_{X|\theta}(x|\theta) = \frac{\theta}{2} \exp\{-\theta|x|\}$, $-\infty < x < \infty, \theta > 0$.

(vi) $f_{X|\theta_1, \theta_2}(x|\theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1}$, $\theta_1 \leq x \leq \theta_2$.

(vii) $f_{X|\theta_1, \theta_2}(x|\theta_1, \theta_2) = \theta_1 \theta_2^{\theta_1} x^{-\theta_1-1}$, $\theta_2 \leq x, \theta_1, \theta_2 > 0$.

- 4 An estimator, T , is an *unbiased* estimator of function $\tau(\theta)$ of parameter θ if

$$E_{f_{T|\theta}}[T] = \tau(\theta)$$

where f_T is the *sampling distribution* of T . The *bias*, $b_\theta(T)$, and *Mean Squared Error*, **MSE**, of an estimator T of $\tau(\theta)$ are defined respectively by

$$b_\theta(T) = E_{f_{T|\theta}}[T] - \tau(\theta)$$

$$\text{MSE}_\theta(T) = E_{f_{T|\theta}}[(T - \tau(\theta))^2]$$

Suppose that $X_1, \dots, X_n$ are a random sample from a $\text{Poisson}(\lambda)$ distribution. Find the maximum likelihood estimator of λ , and show that this estimator is unbiased. Also, find the maximum likelihood estimator of $\tau(\lambda) = e^{-\lambda} = P_\theta[X = 0]$, and find the approximate bias for this estimator using a Taylor expansion.

- 5 Suppose that $X_1, \dots, X_n$ are a random sample from the probability distribution with pdf

$$f_{X|\lambda, \eta}(x|\lambda, \eta) = \lambda e^{-\lambda(x-\eta)} \quad x > \eta$$

and zero otherwise. Find the maximum likelihood estimators of λ and η .

- 6 (From lectures - work through the details) Suppose that $X_1, \dots, X_n$ are a random sample from the probability distribution with pdf

$$f_{X|\theta}(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \quad x > 0.$$

Show that the sample mean $\bar{X}$ is an unbiased estimator of θ . Show also that, if random variable Y_1 is defined as $Y_1 = \min \{X_1, \dots, X_n\}$ then random variable $Z = nY_1$ is also unbiased for θ .

- 7 Suppose that $X_1, \dots, X_n$ are a random sample from a $Uniform(\theta - 1, \theta + 1)$ distribution. Show that the sample mean $\bar{X}$ is an unbiased estimator of θ . Let Y_1 and Y_n be the smallest and largest order statistics derived from $X_1, \dots, X_n$. Show also that random variable $M = (Y_1 + Y_n)/2$ is an unbiased estimator of θ .

- 8 Suppose that $X_1, \dots, X_n$ are a random sample from a $Gamma(2, \lambda)$ distribution.

- (i) Find the maximum likelihood estimator of λ .
- (ii) Find the maximum likelihood estimator, denoted T say, of $\tau = 1/\lambda$.
- (iii) Find $E_{f_{T|\lambda}}[T]$ and $E_{f_{T|\lambda}}[T^2]$.

- 9 Consider the location family pdf with standard member the *Cauchy* distribution

$$f_{X|\theta}(x|\theta) = \frac{1}{\pi} \frac{1}{1 + (x - \theta)^2} \quad -\infty < x < \infty$$

for location parameter $\theta \in \Theta \equiv \mathbb{R}$.

- (a) Derive the *score equation* for θ defined for a random sample $X_1, \dots, X_n$ from this pdf by

$$\frac{\partial l(\theta|\underline{x})}{\partial \theta} = 0$$

where $l(\theta|\underline{x}) = \log L(\theta|\underline{x}) = \log f_{X|\theta}(\underline{x}|\theta)$

- (b) Using a computer package, plot the log-likelihood function $l(\theta|\underline{x})$ for a suitable range of θ for the following observed x values:

7.36 5.14 3.71 3.15 6.00 6.38 1.34 6.73

and hence find the maximum likelihood (ML) estimate.

- 10 Carry out a simulation study to examine the sampling distribution of the maximum likelihood estimator $\hat{\theta}(\underline{X})$ in the Cauchy location family example in the previous problem.

For example, in R:

- Produce $N = 5000$ simulated data sets of size $n = 8$, using a specific value of θ , and using the random number generation function `rcauchy`.
- For each simulated data set, use pointwise evaluation of the likelihood (or the function `optimize`) to evaluate the ML estimate in each case.
- Display using a histogram the distribution of the N stored ML estimates.

The sample median is an alternative estimator of θ . Repeat the computations above using this alternative estimator.