

MATH 557 - EXERCISES 1

These exercises are not for assessment

- 1 Find sufficient statistics for random samples of size n from the following distributions. Note that the sufficient statistics may be multidimensional, but must have dimension no greater than two.

- (a) The Beta density with parameters α and β

$$f_{X|\alpha,\beta}(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad 0 < x < 1$$

and zero otherwise, for $\alpha, \beta > 0$.

- (b) The pmf

$$f_{X|\theta}(x|\theta) = \frac{(\log \theta)^x}{\theta x!} \quad x = 0, 1, 2, \dots$$

and zero otherwise, for $\theta > 1$.

- (c) The Uniform density given by

$$f_{X|\theta}(x|\theta) = \frac{1}{\theta} \quad \theta < x < 2\theta$$

and zero otherwise, for $\theta > 0$.

- 2 Suppose that $X_1, \dots, X_n$ are a random sample from an *Exponential*(λ) distribution,

$$f_{X|\lambda}(x|\lambda) = \lambda e^{-\lambda x} \quad x > 0$$

and zero otherwise, for $\lambda > 0$.

- (a) A sufficient statistic $\underline{T}(\underline{X})$ is termed a *minimal sufficient statistic* if, for any other sufficient statistic, $\underline{T}'(\underline{X})$, $\underline{T}(\underline{x})$ is a function of $\underline{T}'(\underline{x})$; specifically, for $\underline{x}, \underline{x}^* \in \mathcal{X}$, if $\underline{T}(\underline{X})$ is minimal sufficient

$$\underline{T}'(\underline{x}) = \underline{T}'(\underline{x}^*) \implies \underline{T}(\underline{x}) = \underline{T}(\underline{x}^*)$$

Derive a minimal sufficient statistic for λ .

- (b) Suppose now that only the m smallest of the X_i s are observed. Derive a sufficient statistic for λ based on this reduced sample of size m .

Hint: Construct the joint density of the first m order statistics $X_{(1)}, \dots, X_{(m)}$ by noting that

$$X_{(m)} = x \iff X_{(r)} > x, \text{ for all } r > m.$$

- 3 Suppose that $X_1, \dots, X_n$ are a random sample from a *Poisson*(θ) distribution. It can be shown that

$$T(\underline{X}) = \sum_{i=1}^n X_i$$

is a sufficient statistic for θ . Find the conditional mass function of $\underline{X}$ given that $T(\underline{X}) = t$.

4 Let $\underline{X} = (X_1, \dots, X_n)$ be a random vector with joint density in $\mathcal{F}_k = \{f_i(\underline{x}) : i = 0, \dots, k\}$, so that $\mathcal{F}_k$ is parameterized by index $i \in \{0, \dots, k\}$. Assume that the densities in $\mathcal{F}_k$ have common support.

(a) Show that the statistic

$$T(\underline{X}) = \left(\frac{f_1(\underline{X})}{f_0(\underline{X})}, \dots, \frac{f_k(\underline{X})}{f_0(\underline{X})} \right)^T$$

is minimal sufficient for i .

(b) Let $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ be a family of densities with common support, and suppose that

- $f_i \equiv f_{\theta_i} \in \mathcal{F}$, θ_i distinct, for $i = 0, \dots, k$ and
- $T(\underline{X})$ as defined in part (a) is sufficient for θ .

Show that $T(\underline{X})$ is minimal sufficient for θ .

(c) Show that 1-1 functions of minimal sufficient statistics are minimal sufficient statistics.

(d) Use the results from parts (b) and (c) to show that the sample mean $\bar{X}$ is minimal sufficient for β if $\underline{X}$ is a random sample from an Exponential distribution with expectation $\beta > 0$.

5 Suppose that $X_1, \dots, X_n$ are a random sample from a *Multinomial*(3, $\underline{\theta}$) distribution defined by the probabilities

$$\Pr[X_i = j] = \theta_j \quad j = 1, 2, 3$$

and zero otherwise, where $0 < \theta_1, \theta_2, \theta_3 < 1$, and $\theta_1 + \theta_2 + \theta_3 = 1$.

(a) Find a (possibly vector-valued) sufficient statistic for $\underline{\theta}$.

(b) Find the (joint) pmf of the sufficient statistic.

(c) Find the form of the maximum likelihood estimator for $\underline{\theta}$