

MATH 557 - ASSIGNMENT 3 SOLUTIONS

- 1 By inspection $\tau(\theta) = e^{-\theta}$. Recall that $T = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ and also for $\tau(\theta)$. Therefore, by the Corollary from notes, the unique best unbiased estimator of τ is a function of T , $h(T)$ say. For unbiasedness, we must have

$$E_{f_{T|\theta}}[h(T)] = \sum_{t=0}^{\infty} h(t) \frac{(n\theta)^t e^{-n\theta}}{t!} = e^{-\theta}$$

so therefore if and only if $h(t) = ((n-1)/n)^t$

$$\sum_{t=0}^{\infty} h(t) \frac{(n\theta)^t e^{-n\theta}}{t!} = \sum_{t=0}^{\infty} \left(\frac{n-1}{n}\right)^t \frac{(n\theta)^t e^{-n\theta}}{t!} = e^{-n\theta} \sum_{t=0}^{\infty} \frac{((n-1)\theta)^t}{t!} = e^{-\theta}$$

Therefore the best unbiased estimator is

$$h(T) = \left(\frac{n-1}{n}\right)^T$$

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- 2 (a) The likelihood is

$$L(\theta|\underline{x}) = \prod_{i=1}^n \frac{2\theta^2}{x_i^3} \mathbb{1}_{(\theta, \infty)}(x_i) = \frac{2^n}{h(\underline{x})} \theta^{2n} \mathbb{1}_{(\theta, \infty)}(x_{(1)}) \quad \text{where } h(\underline{x}) = \prod_{i=1}^n x_i^3$$

This is monotonic increasing on $(0, x_{(1)})$, and zero elsewhere. To respect the fact that the parameter space is the interval $(0, 1]$, the ML estimator is $\hat{\theta}_{ML} = \min\{X_{(1)}, 1\} = M$, say. Now, the cdf of $S = X_{(1)}$ is

$$f_{S|\theta}(s|\theta) = \frac{dF_{S|\theta}(s|\theta)}{ds} = \frac{d}{ds} \left\{ 1 - \left(\frac{\theta}{s}\right)^{2n} \right\} = \frac{2n\theta^{2n}}{s^{2n+1}} \quad s > \theta$$

and therefore

$$\begin{aligned} E_{f_{M|\theta}}[M^r] &= \int_{\theta}^1 s^r \frac{2n\theta^{2n}}{s^{2n+1}} ds + \int_1^{\infty} \frac{2n\theta^{2n}}{s^{2n+1}} ds = 2n\theta^{2n} \left[-\frac{1}{2n-r} \frac{1}{s^{2n-r}} \right]_{\theta}^1 + 2n\theta^{2n} \left[-\frac{1}{2n} \frac{1}{s^{2n}} \right]_1^{\infty} \\ &= \frac{2n\theta^{2n}}{2n-r} \left(\frac{1}{\theta^{2n-r}} - 1 \right) + \theta^{2n} = \frac{2n}{2n-r} (\theta^r - \theta^{2n}) + \theta^{2n} \end{aligned}$$

so therefore

$$E_{f_{M|\theta}}[M] = \frac{2n}{2n-1} (\theta - \theta^{2n}) + \theta^{2n} = \frac{2n\theta}{2n-1} - \frac{\theta^{2n}}{2n-1}$$

and the bias of S is

$$b_S(\theta) = \frac{2n\theta}{2n-1} - \frac{2n\theta^{2n}}{2n-1} - \theta = \frac{\theta(1 - \theta^{2n-1})}{2n-1}$$

(that is, positive as $0 < \theta \leq 1$, but tending to zero as $n \rightarrow \infty$).

The variance of the estimator is

$$\text{Var}_{f_{M|\theta}}[M] = E_{f_{M|\theta}}[M^2] - \{E_{f_{M|\theta}}[M]\}^2 = \frac{2n}{2n-2} (\theta^2 - \theta^{2n}) + \theta^{2n} - \left(\frac{2n\theta}{2n-1} - \frac{\theta^{2n}}{2n-1} \right)^2$$

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(b) Let $T = T(S)$ take the form

$$T(S) = aS\mathbb{1}_{(0,1]}(S) + b\mathbb{1}_{(1,\infty]}(S)$$

The expected value of T is

$$\begin{aligned} E_{f_{T|\theta}}[T] &= E_{f_{S|\theta}}[T(S)] = \int_{\theta}^{\infty} (as\mathbb{1}_{(0,1]}(s) + b\mathbb{1}_{(1,\infty]}(s)) \frac{2n\theta^{2n}}{s^{2n+1}} ds \\ &= 2an\theta^{2n} \int_{\theta}^1 \frac{1}{s^{2n}} ds + 2bn\theta^{2n} \int_1^{\infty} \frac{1}{s^{2n+1}} ds \\ &= \frac{2an\theta^{2n}}{2n-1} \left(\frac{1}{\theta^{2n-1}} - 1 \right) - \frac{2bn\theta^{2n}}{2n} \\ &= \frac{2an(\theta - \theta^{2n})}{2n-1} - b\theta^{2n} \end{aligned}$$

Thus T is unbiased if this expectation is equal to θ for all θ , that is

$$\frac{2an(\theta - \theta^{2n})}{2n-1} - b\theta^{2n} = \theta$$

Equating coefficients, we must have

$$\frac{2an}{2n-1} = 1 \quad \frac{2an}{2n-1} - b = 0 \quad \implies \quad a = \frac{2n-1}{2n} = 1 - \frac{1}{2n} \quad b = 1$$

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(c) Consider $U(S)$ such that $E[U(S)] = 0$ for all θ . We must have

$$E[U(S)] = \int_{\theta}^{\infty} U(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds = 0$$

or, equivalently

$$\int_{\theta}^1 \frac{U(s)}{s^{2n+1}} ds + \int_1^{\infty} \frac{U(s)}{s^{2n+1}} ds = 0.$$

As this must hold for all $\theta \in (0, 1]$, we must have $U(s) \equiv 0$ for $s \in (0, 1]$, and also

$$\int_1^{\infty} \frac{U(s)}{s^{2n+1}} ds = 0. \tag{1}$$

Now, plugging in the proposed form of $T(S)$, we have

$$\begin{aligned} \text{Cov}_{f_{T,U}}[T, U] &= E_{f_{S|\theta}}[T(S)U(S)] = \int_{\theta}^{\infty} T(s)U(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds \\ &= \int_{\theta}^{\infty} (as\mathbb{1}_{(0,1]}(s) + b\mathbb{1}_{(1,\infty]}(s))U(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds \\ &= a \int_{\theta}^1 sU(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds + b \int_1^{\infty} U(s) \frac{2n\theta^{2n}}{s^{2n+1}} ds = 0 \end{aligned}$$

as $U(s)$ is zero on the domain of the first integral, and the second integral is zero, by the earlier result in equation (1). Thus, by the theorem from lectures, as the estimator $T(S)$ is uncorrelated with all unbiased estimators of zero, then it is best unbiased.

We can also demonstrate directly that T is a function of a complete (minimal) sufficient statistic - completeness can be proved directly using calculus methods - and hence conclude that T is best unbiased.

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3 We have

$$\tau(\theta) = \frac{\theta(3+2\theta)}{6} = \frac{1}{2}\theta + \frac{1}{3}\theta^2$$

By previous results, $T = X_{(n)} = \max\{X_1, \dots, X_n\}$ is minimal sufficient for θ . By results for maximum order statistics

$$f_{T|\theta}(t|\theta) = \frac{nt^{n-1}}{\theta^n} \mathbb{1}_{(0,\theta)}(t)$$

Therefore

$$E_{f_{T|\theta}}[g(T)] = \int_0^\theta g(t) \frac{nt^{n-1}}{\theta^n} dt = \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt.$$

If this is zero for all θ , then g must be zero almost everywhere, so T is also complete. Now,

$$E_{f_{T|\theta}}[T^r] = \int_0^\theta t^r \frac{nt^{n-1}}{\theta^n} dt = \frac{n}{\theta^n} \int_0^\theta t^{n+r-1} dt = \frac{n}{\theta^n} \left[\frac{t^{n+r}}{n+r} \right]_0^\theta = \frac{n\theta^r}{n+r}$$

so

$$E_{f_{T|\theta}}[T] = \frac{n\theta}{n+1} \quad E_{f_{T|\theta}}[T^2] = \frac{n\theta^2}{n+2}$$

Therefore, the estimator

$$\hat{\theta}(T) = \frac{(n+1)}{2n}T + \frac{(n+2)}{3n}T^2$$

is the best unbiased estimator, as it is a function of the complete sufficient statistic T .

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