MATH 557 - ASSIGNMENT 1
SOLUTIONS

1 Let $\mathcal{N}_\theta = \{0, 1, \ldots, \theta\}$. For two vectors $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, and any fixed $\theta \in \mathbb{N}$,

$$\frac{f_{\underset{\sim}{X}|\theta}(\underset{\sim}{x}|\theta)}{f_{\underset{\sim}{Y}|\theta}(\underset{\sim}{y}|\theta)} = \frac{\prod_{i=1}^{n} \mathbb{1}_{\mathcal{N}_\theta}(x_i)}{\prod_{i=1}^{n} \mathbb{1}_{\mathcal{N}_\theta}(y_i)} K(\underset{\sim}{x}, \underset{\sim}{y}) \theta^{(s_x - s_y)} \tag{1}$$

where

$$s_x = \sum_{i=1}^{n} x_i \qquad s_y = \sum_{i=1}^{n} y_i \qquad K(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\prod_{i=1}^{n} y_i!}{\prod_{i=1}^{n} x_i!}$$

and $\mathbb{1}_A(x)$ is the indicator function for set A. The ratio in (1) does not depend on θ (for **any** value of θ) if and only if $s_x = s_y$ and the ratio of indicator functions equals 1, which occurs if

$$X_{(n)} = \max\{x_1, \ldots, x_n\} \leq \theta \quad \text{and} \quad Y_{(n)} = \max\{y_1, \ldots, y_n\} \leq \theta$$

Thus, by the Theorem from lectures, $\underset{\sim}{T}(\underset{\sim}{X}) = \left(\sum_{i=1}^{n} X_i, X_{(n)} \right)$ is a minimal sufficient statistic.

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2 Let $Z = 0$ correspond to Tails, $Z = 1$ correspond to Heads.

(a) At the end of the study, we have **observed** the values of random variables, M, N, Z and $X_1, \ldots, X_N$, that is, we have observed $M = m$, $N = t$, $Z = z$, and $X_i = x_i$ for $i = 1, \ldots, t$. By independence, we have the full joint distribution as

$$f_Z(z) f_{N|Z,\theta}(t|z,\theta) f_{M|Z,N,\theta}(m|z,t,\theta) f_{X_1,\ldots,X_t|N,M,Z,\theta}(x_1, \ldots, x_t|t,m,z,\theta)$$

in which $f_Z(z) = 1/2$, $z = 0, 1$, zero otherwise. Furthermore

$$f_{N|Z,\theta}(t|z,\theta) = \{\mathbb{1}_{\{n\}}(t)\}^z \left\{ \binom{k-1}{t-1} \theta^k (1-\theta)^{n-k} \right\}^{1-z}$$

(as, given $z = 0$, the distribution of N given Z is negative binomial, whereas given $z = 1$, the distribution of N is degenerate at n). Secondly

$$f_{M|Z,N,\theta}(m|z,t,\theta) = \left\{ \binom{t}{m} \theta^m (1-\theta)^{t-m} \right\}^z \left\{ \mathbb{1}_{\{k\}}(m) \right\}^{1-z}$$

Finally,

$$f_{X_1,\ldots,X_t|N,M,Z,\theta}(x_1, \ldots, x_t|t,m,z,\theta) = \binom{t}{m}^{-1}$$

as conditional on $N = t$ and $M = m$, by the sufficiency result from lectures, the conditional distribution of $X_1, \ldots, X_t$ is uniform on the set of binary sequences of length t that contains m 1s.

Thus the full joint distribution is

$$\frac{1}{2}\{\mathbb{1}_{\{n\}}(t)\}^z \left\{\binom{k-1}{t-1}\theta^k(1-\theta)^{n-k}\right\}^{1-z} \left\{\binom{t}{m}\theta^m(1-\theta)^{t-m}\right\}^z \left\{\mathbb{1}_{\{k\}}(m)\right\}^{1-z} \binom{t}{m}^{-1}$$

which can be re-written

$$C_1(k,n,t,m,z) \times \{\mathbb{1}_{\{n\}}(t)\}^z \left\{\mathbb{1}_{\{k\}}(m)\right\}^{1-z} \theta^{k(1-z)+mz}(1-\theta)^{(1-z)(n-k)+z(t-m)} \tag{2}$$

where

$$C_1(k,n,t,m,z) = \frac{1}{2}\binom{k-1}{t-1}^{1-z}\binom{t}{m}^z\binom{t}{m}^{-1}.$$

Now, if $z = 0$, the second term in (2) equals

$$\left\{\mathbb{1}_{\{k\}}(m)\right\}\theta^k(1-\theta)^{n-k}$$

which equals $\theta^m(1-\theta)^{n-m}$ if $m = k$, and zero otherwise. If $z = 1$, the second term in (2) equals

$$\{\mathbb{1}_{\{n\}}(t)\}\theta^k(1-\theta)^{t-m}$$

which equals $\theta^m(1-\theta)^{n-m}$ if $t = n$, and zero otherwise. Therefore, the second term equals

$$\theta^m(1-\theta)^{n-m}$$

if $(z = 0, m = k)$ or $(z = 1, t = n)$, and zero otherwise, and we can rewrite the full joint distribution as

$$C_1(k,n,t,m,z)\theta^m(1-\theta)^{n-m}$$

if $(z = 0, m = k)$ or $(z = 1, t = n)$, and zero otherwise. Hence, by the Neyman factorization theorem, the random variables (N, M) jointly form a sufficient statistic for θ.

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(b) The conditional distribution of Z given N and M can be obtained by dividing the joint distribution of (Z, N, M) by the marginal for (N, M). The marginal of N, M is obtained from above as

$$\frac{1}{2}\mathbb{1}_{\{n\}}(t)\binom{t}{m}\theta^m(1-\theta)^{t-m} + \frac{1}{2}\mathbb{1}_{\{k\}}(m)\binom{k-1}{t-1}\theta^k(1-\theta)^{n-k} \tag{3}$$

which is zero if both $t \neq n$ and $m \neq k$. If this is **not** the case, then the term in (3) is proportional to $\theta^m(1-\theta)^{n-m}$, with constant of proportionality

$$C_2(k,n,t,m) = \begin{cases} \frac{1}{2}\binom{n}{m} & t = n, m \neq k \\ \frac{1}{2}\binom{k-1}{t-1} & t \neq n, m = k \\ \frac{1}{2}\binom{n}{k} + \frac{1}{2}\binom{k-1}{n-1} & t = n, m = k \end{cases}$$

Therefore, the conditional distribution of Z given $N = t$ and $M = m$ is

$$\frac{C_1(k,n,t,m,z)}{C_2(k,n,t,m)}$$

with the denominator non-zero, which does not depend on θ.

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3 (a) The joint pmf for data $y_1, \ldots, y_n$ is

$$\begin{aligned} f_{\underset{\sim}{Y}|\underset{\sim}{\beta}}(\underset{\sim}{y}|\underset{\sim}{\beta}) &= \prod_{i=1}^{n} \binom{m_i}{y_i} \psi_i(\underset{\sim}{\beta})^{y_i}(1-\psi_i(\underset{\sim}{\beta}))^{m_i-y_i} \\ &= \prod_{i=1}^{n} \binom{m_i}{y_i} \frac{\exp\{y_i(\beta_0+\beta_1 x_i)\}}{(1+\exp\{\beta_0+\beta_1 x_i\})^{m_i}} \\ &= \frac{\prod_{i=1}^{n} \binom{m_i}{y_i}}{\prod_{i=1}^{n}(1+\exp\{\beta_0+\beta_1 x_i\})^{m_i}} \exp\left\{\beta_0 \sum_{i=1}^{n} y_i + \beta_1 \sum_{i=1}^{n} x_i y_i\right\} \end{aligned}$$

Therefore, for two data vectors $\underset{\sim}{y}, \underset{\sim}{y}^*$

$$\frac{f_{\underset{\sim}{Y}|\underset{\sim}{\beta}}(\underset{\sim}{y}|\underset{\sim}{\beta})}{f_{\underset{\sim}{Y}|\underset{\sim}{\beta}}(\underset{\sim}{y}^*|\underset{\sim}{\beta})} = \frac{\prod_{i=1}^{n} \binom{m_i}{y_i}}{\prod_{i=1}^{n} \binom{m_i}{y_i^*}} \exp\left\{\beta_0\left(\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i^*\right) + \beta_1\left(\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i y_i^*\right)\right\}$$

This ratio does not depend on $\underset{\sim}{\beta}$ if and only if

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_i^* \qquad \text{and} \qquad \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i y_i^*$$

Therefore

$$\underset{\sim}{T}(\underset{\sim}{Y}) = \left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} x_i Y_i\right)$$

is a minimal sufficient statistic.

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(b) From the formula, denoting by $\psi(x)$ the probability ψ as an explicit function of x

$$\psi(0) = \frac{\exp\{\beta_0\}}{1+\exp\{\beta_0\}} \qquad \psi(1) = \frac{\exp\{\beta_0+\beta_1\}}{1+\exp\{\beta_0+\beta_1\}}$$

so therefore

$$\frac{\psi(0)}{1-\psi(0)} = \exp\{\beta_0\} \qquad \frac{\psi(1)}{1-\psi(1)} = \exp\{\beta_0+\beta_1\}$$

and

$$\frac{\psi(1)/(1-\psi(1))}{\psi(0)/(1-\psi(0))} = \exp\{\beta_1\} \tag{4}$$

Recalling that $\psi_i(x)$ is the success probability for the i experiment, for $x = 0, 1$, the quantity

$$\frac{\psi(x)}{1-\psi(x)}$$

is the **odds on success** for that experiment. Thus, from (4), we see that β_1 measures the **odds ratio** for success on the log scale when $x = 1$ and $x = 0$ for every experiment.

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