

MATH 557 - ASSIGNMENT 2

To be handed in not later than 5pm, 16th February 2010.

Please hand in during lectures, to Burnside 1225, or to the Mathematics Office Burnside 1005

Suppose that data $x_1, \dots, x_n$ arising from a random sample $X_1, \dots, X_n$ from an $Exponential(\lambda)$ distribution are available. Here the Exponential distribution is parameterized

$$f_{X|\lambda}(x|\lambda) = \lambda e^{-\lambda x} \quad x > 0$$

and zero otherwise. Note that all estimators $\hat{\lambda}$ are functions of $\underline{X}$, that is $\hat{\lambda} \equiv \hat{\lambda}(\underline{X})$.

- (a) Find the maximum likelihood (ML) and method of moments (MM) estimators of λ , denoted $\hat{\lambda}_{ML}$ and $\hat{\lambda}_{MM}$ respectively.

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- (b) Find the Bayes estimator of λ , $\hat{\lambda}_B$, assuming an $Exponential(\alpha)$ prior for λ ($\alpha > 0$ is some fixed constant) under the squared-error loss function

$$\mathcal{L}(\hat{\lambda}_B, \lambda) = (\hat{\lambda}_B - \lambda)^2$$

3 MARKS

- (c) Find the minimum mean-square (MS) estimator of λ , denoted $\hat{\lambda}_{MS}$ and defined by

$$\hat{\lambda}_{MS} = \underset{\hat{\lambda}}{\operatorname{argmin}} E_{f_{\underline{X}|\lambda}}[(\hat{\lambda} - \lambda)^2]$$

3 MARKS

- (d) For the estimators $\hat{\lambda}_{ML}$, $\hat{\lambda}_{MM}$ and $\hat{\lambda}_B$ from (a) and (b), comment on the relative sizes of the variances for fixed n , and as $n \rightarrow \infty$.

*Note: Here consider the **frequentist** variances of the estimators, that is, the variances of the sampling distributions of the estimators given the true value of λ . I will accept simulation-based arguments where analytical results are not available.*

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- (e) For the Bayesian model in (b), find the *prior predictive* distribution for $\underline{X}$ defined by

$$f_{\underline{X}|\alpha}(\underline{x}|\alpha) = \int_0^\infty f_{\underline{X}|\lambda}(\underline{x}|\lambda) \pi_{\lambda|\alpha}(\lambda|\alpha) d\lambda$$

where $\pi_{\lambda|\alpha}(\lambda|\alpha)$ is the prior distribution for λ .

3 MARKS

- (f) For the Bayesian model in (b), find the *posterior predictive* distribution for another set of random variables $\underline{X}^* = (X_1^*, \dots, X_m^*)$, presumed to be $Exponential(\lambda)$ distributed and conditionally independent given λ , in light of data $\underline{X} = \underline{x}$ and the specified model, defined by

$$f_{\underline{X}^*|\underline{X},\alpha}(\underline{x}^*|\underline{x},\alpha) = \int_0^\infty f_{\underline{X}^*|\lambda}(\underline{x}^*|\lambda) \pi_{\lambda|\underline{X},\alpha}(\lambda|\underline{x},\alpha) d\lambda$$

where $\pi_{\lambda|\underline{X},\alpha}(\lambda|\underline{x},\alpha)$ is the posterior distribution for λ implied by the model in light of the data.

Comment on the behaviour of the posterior predictive as $m \rightarrow \infty$.

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