MATH 556 – FALL 2022
MID-TERM EXAMINATION

Marks can be obtained by answering all questions. All questions carry equal marks.

The total mark available is 60, but rescaling of the final mark may occur.

Show your working in all cases. You may quote results from the formula sheet or from lecture notes if you make explicit the result you are quoting.

1. Identify which of the following functions are legitimate specifications for pmfs or pdfs (denoted f) or cdfs (denoted F) for a scalar random variable, giving reasons as to your conclusions. Note that $\mathbb{1}_A(x)$ is the indicator function for set A that takes the value 1 if $x \in A$ and zero otherwise.

   (a) For some constant c, the function
   $$f(x) = \frac{ce^{-x}}{(1+e^{-x})^3} \qquad x \in \mathbb{R}$$
   is proposed as a pdf. 5 MARKS

   (b) The function
   $$F(x) = \frac{1}{2}\mathbb{1}_{\{0\}}(x) + \frac{1}{2}\mathbb{1}_{[0,\infty)}(x)(1-e^{-x}) \qquad x \in \mathbb{R}$$
   is proposed as a cdf. 5 MARKS

   (c) For some constant c, the function
   $$f(x) = c\mathbb{1}_{\mathbb{X}}(x)\frac{1}{x!}$$
   is proposed as a pmf with support $\mathbb{X} \equiv \{1, 2, \ldots\}$. 5 MARKS

2. (a) Suppose X is continuous with cdf
   $$F_X(x) = \frac{e^{x+1}}{1+e^{x+1}} \qquad x \in \mathbb{R}$$
   Find the quantile function for X, $Q_X(p)$ for $0 < p < 1$. 5 MARKS

   (b) Suppose Y is continuous with cdf
   $$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y^2}{4+y^2} & y > 0 \end{cases}$$
   Find the quantile function for Y, $Q_Y(p)$ for $0 < p < 1$. 5 MARKS

   (c) Find a transformation $g(\cdot)$ such that for variable X from part (a), $g(X)$ has the same distribution as Y from part (b).

   5 MARKS

3. Suppose that X_1 and X_2 have a joint pdf given by

$$f_{X_1,X_2}(x_1,x_2) = c\mathbb{1}_{\mathbb{X}}(x_1,x_2)\exp\{-x_1\} \qquad (x_1,x_2) \in \mathbb{R}^2$$

for some constant c, where $\mathbb{X} = \{(x_1,x_2) : 0 < x_2 < x_1 < \infty\}$ and $\mathbb{1}_A(x_1,x_2)$ is the indicator function for set $A \subseteq \mathbb{R}^2$.

(a) Find the value of c. 4 MARKS

(b) Find the marginal pdf of X_1. 4 MARKS

(c) Find the conditional pdf of X_2 given that $X_1 = x_1$, where $x_1 > 0$. 4 MARKS

(d) Are X_1 and X_2 independent ? Justify your answer. 3 MARKS

4. Suppose that X_1 and X_2 are independent $Exponential(1)$ random variables.

(a) Find $\mathbb{E}_{X_1}[X_1^3]$. 4 MARKS

(b) Find

$$\mathbb{E}_{X_1,X_2}[X_1^3 + X_2^3] = \int_0^\infty \int_0^\infty (x_1^3 + x_2^3) f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2.$$

where $f_{X_1,X_2}(x_1,x_2)$ is the joint pdf of X_1 and X_2. 5 MARKS

(c) For $y > 0$, find

$$P_{X_1,X_2}[X_1 < yX_2]$$

6 MARKS

Hint: this probability is given by

$$\iint_{A_y} f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2$$

where for fixed $y > 0$,

$$A_y = \{(x_1,x_2) : x_1 < yx_2\}.$$