MATH 556 – FALL 2022
MID-TERM EXAMINATION: SOLUTIONS

1. *Identify which of the following functions are legitimate specifications for pmfs or pdfs (denoted f) or cdfs (denoted F) for a scalar random variable, giving reasons as to your conclusions. Note that $\mathbb{1}_A(x)$ is the indicator function for set A that takes the value 1 if $x \in A$ and zero otherwise.*

   (a) *For some constant c, the function*

   $$f(x) = \frac{ce^{-x}}{(1+e^{-x})^3} \qquad x \in \mathbb{R}$$

   is proposed as a pdf.

   This is a pdf as it is non-negative and integrable on $\mathbb{R}$; in fact

   $$\int_{-\infty}^{\infty} \frac{e^{-x}}{(1+e^{-x})^3}\, dx = \left[-\frac{1}{2(1+e^{-x})^2} \right]_{-\infty}^{\infty} = \frac{1}{2}$$

   so $c = 2$. 5 MARKS

   (b) *The function*

   $$F(x) = \frac{1}{2}\mathbb{1}_{\{0\}}(x) + \frac{1}{2}\mathbb{1}_{[0,\infty)}(x)(1 - e^{-x}) \qquad x \in \mathbb{R}$$

   is proposed as a cdf.

   This is a not a cdf as

   $$\lim_{x \longrightarrow \infty} = \frac{1}{2}$$

   5 MARKS

   (c) *For some constant c, the function*

   $$f(x) = c\mathbb{1}_{\mathbb{X}}(x)\frac{1}{x!}$$

   is proposed as a pmf with support $\mathbb{X} \equiv \{1, 2, \ldots\}$. 5 MARKS

   This is a valid pmf: it is a positive function on $\mathbb{X}$, and as the $Poisson(1)$ pmf takes the form

   $$e^{-1}\frac{1}{x!} \qquad x \in \{0, 1, 2, \ldots\}$$

   the function is clearly also summable. In fact

   $$\sum_{x=1}^{\infty} \frac{1}{x!} = \sum_{x=0}^{\infty} \frac{1}{x!} - 1 = e - 1$$

   using the Poisson pmf sum. Thus $c = 1/(e-1) \simeq 0.5819767$.

2. (a) *Suppose X is continuous with cdf*

$$F_X(x) = \frac{e^{x+1}}{1+e^{x+1}} \qquad x \in \mathbb{R}$$

Find the quantile function for X, $Q_X(p)$ for $0 < p < 1$. 5 MARKS

In this case, we may solve

$$p = \frac{e^{x+1}}{1+e^{x+1}} \qquad \Longleftrightarrow \qquad x = \log\left(\frac{p}{1-p}\right) - 1$$

so therefore $0 < p < 1$

$$Q_X(p) = \log\left(\frac{p}{1-p}\right) - 1 = \text{logit}(p) - 1$$

say.

(b) *Suppose Y is continuous with cdf*

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y^2}{4+y^2} & y > 0 \end{cases}$$

Find the quantile function for Y, $Q_Y(p)$ for $0 < p < 1$. 5 MARKS

Here we solve

$$p = \frac{y^2}{4+y^2} \qquad \Longleftrightarrow \qquad y = \sqrt{\frac{4p}{(1-p)}}$$

so therefore for $0 < p < 1$

$$Q_Y(p) = \sqrt{\frac{4p}{(1-p)}} = 2\sqrt{\frac{p}{(1-p)}}.$$

(c) *Find a transformation $g(\cdot)$ such that for variable X from part (a), $g(X)$ has the same distribution as Y from part (b).*

The transformation we need is

$$Y = Q_Y(F_X(X)) = 2\sqrt{\exp\{X+1\}}$$

as then, for $y > 0$,

$$\begin{aligned} F_Y(y) = P_Y[Y \le y] &= P_X[2\sqrt{\exp\{X+1\}} \le y] \\ &= P_X[\exp\{X+1\} \le y^2/4] \\ &= P_X[X \le \log(y^2/4) - 1] \\ &= \frac{y^2/4}{1+y^2/4} \end{aligned}$$

5 MARKS

3. *Suppose that* X_1 *and* X_2 *have a joint pdf given by*

$$f_{X_1,X_2}(x_1,x_2) = c\mathbb{1}_{\mathbb{X}}(x_1,x_2)\exp\{-x_1\}$$

where $\mathbb{X} = \{(x_1,x_2) : 0 < x_2 < x_1 < \infty\}$*, for some constant* c

Note that we may write

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2|X_1}(x_2|x_1) = cx_1\exp\{-x_1\}\frac{1}{x_1} \qquad 0 < x_2 < x_1 < \infty$$

and deduce that

$$f_{X_1}(x_1) = x_1\exp\{-x_1\} \qquad x_1 > 0$$

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1} \qquad 0 < x_2 < x_1$$

so $X_1 \sim Gamma(2,1)$ and $X_2|X_1 = x_1 \sim Uniform(0,x_1)$.

(a) *Find the value of* c*.*

From the above logic, we have $c = 1$. 4 MARKS

(b) *Find the marginal pdf of* X_1*.*

For $x_1 > 0$, we have

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_2 = \int_0^{x_1}\exp\{-x_1\}\,dx_2 = x_1\exp\{-x_1\}.$$

4 MARKS

(c) *Find the conditional pdf of* X_2 *given that* $X_1 = x_1$*, where* $x_1 > 0$*.*

We have that for $0 < x_2 < x_1$, when $x_1 > 0$,

$$f_{X_2|X_1}(x_2|x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} = \frac{1}{x_1}.$$

4 MARKS

(d) *Are* X_1 *and* X_2 *independent ? Justify your answer.*

X_1 and X_2 are not independent as the joint pdf does not factorize into a product of a function of x_1 and a function of x_2 due to the indicator function.

3 MARKS

4. *Suppose that X_1 and X_2 are independent $Exponential(1)$ random variables.*

(a) *Find $\mathbb{E}_{X_1}[X_1^3]$.* 4 MARKS

$$\mathbb{E}_{X_1}[X_1^3] = \int_0^\infty x_1^3 \exp\{-x_1\}\, dx_1 = \Gamma(4) = 6$$

using the fact that the integrand is the kernel of a $Gamma(4,1)$ pdf.

(b) *Find*

$$\mathbb{E}_{X_1,X_2}[X_1^3 + X_2^3] = \int_0^\infty \int_0^\infty (x_1^3 + x_2^3) f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2.$$

where $f_{X_1,X_2}(x_1,x_2)$ is the joint pdf of X_1 and X_2. 5 MARKS

By direct calculation

$$\begin{aligned}
\mathbb{E}_{X_1,X_2}[X_1^3 + X_2^3] &= \int_0^\infty \int_0^\infty (x_1^3 + x_2^3) f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2 \\
&= \int_0^\infty \int_0^\infty (x_1^3 + x_2^3) \exp\{-x_1\} \exp\{-x_2\}\, dx_1 dx_2 \qquad \text{independence} \\
&= \int_0^\infty x_1^3 \exp\{-x_1\}\, dx_1 + \int_0^\infty x_2^3 \exp\{-x_2\}\, dx_2 \\
&= 12.
\end{aligned}$$

Can also use the general result that

$$\mathbb{E}_{X_1,X_2}[X_1^3 + X_2^3] = \mathbb{E}_{X_1}[X_1^3] + \mathbb{E}_{X_2}[X_2^3].$$

irrespective of independence.

(c) *For $y > 0$, find*

$$P_{X_1,X_2}[X_1 < yX_2]$$

6 MARKS

This probability is given by

$$\begin{aligned}
P_{X_1,X_2}[X_1 < yX_2] &= \int_0^\infty \int_0^{yx_2} \exp\{-x_1\} \exp\{-x_2\}\, dx_1 dx_2 \\
&= \int_0^\infty (1 - \exp\{-yx_2\}) \exp\{-x_2\}\, dx_2 \\
&= \left[-\exp\{-x_2\} + \frac{1}{1+y} \exp\{-(y+1)x_2\} \right]_0^\infty \\
&= 1 - \frac{1}{(y+1)} \\
&= \frac{y}{1+y}
\end{aligned}$$