556: MATHEMATICAL STATISTICS I

ONE-SIDED CHEBYCHEV INEQUALITY

In the following theorem, P and $\mathbb{E}$ refer to probabilities and expectations under the probability associated with random variable X.

Theorem (One-Sided Chebychev Inequality)
For random variable X with expectation μ and variance σ^2, and $t > 0$

$$P[X - \mu \geq t] \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

so, setting $t = k\sigma$,

$$P[X - \mu \geq k\sigma] \leq \frac{1}{1 + k^2}$$

Proof Let $t > 0$. If $P[X - \mu \geq t] = 0$ then the result is clearly true, otherwise, let

$$a = P[X - \mu \geq t] \qquad b = \mathbb{E}[X - \mu | X - \mu \geq t].$$

Clearly, for any random variable Z,

$$\mathbb{E}[Z|Z \geq t] \geq t$$

and thus $b \geq t \geq 0$. Hence

$$\begin{aligned} 0 = \mathbb{E}[X - \mu] &= \mathbb{E}[X - \mu | X - \mu \geq t]P[X - \mu \geq t] + \mathbb{E}[X - \mu | X - \mu < t]P[X - \mu < t] \\ &= ba + \mathbb{E}[X - \mu | X - \mu < t](1 - a) \end{aligned}$$

and we have that

$$\mathbb{E}[X - \mu | X - \mu < t] = -\frac{ab}{1 - a}.$$

Hence

$$\begin{aligned} \sigma^2 = \text{Var}[X] &= a\mathbb{E}[(X - \mu)^2 | X - \mu \geq t] + (1 - a)\mathbb{E}[(X - \mu)^2 | X - \mu < t] \\ &= a\mathbb{E}[(X - \mu - b)^2 | X - \mu \geq t] + ab^2 \\ &\quad + (1 - a)\mathbb{E}\left[\left(X - \mu + \frac{ab}{1 - a}\right)^2 \middle| X - \mu < t\right] + (1 - a)\left(\frac{ab}{1 - a}\right)^2 \\ &\geq ab^2 + \frac{(1 - a)a^2b^2}{(1 - a)^2} = \frac{ab^2}{1 - a} \end{aligned} \tag{1}$$

as the two expectation are non-negative. Thus

$$\sigma^2 \geq \frac{ab^2}{1 - a} \geq \frac{at^2}{1 - a} \tag{2}$$

Rearranging, we have

$$P[X - \mu \geq t] = a \leq \frac{\sigma^2}{\sigma^2 + t^2}$$

Equality is achieved in line (1) if the two expectations are zero, that is, if the mass function of X is non-zero only at the two points $\mu + b$ and $\mu - ab/(1 - a)$, and in line (2) when $b = t$. Thus we have equality only when the distribution has at most two non-zero masses, at $\mu + t$ and $\mu - at/(1 - a)$.