

MATH 556 - ASSIGNMENT 2: SOLUTIONS

1 By direct calculation, we have by the theorem of total probability for $y \geq 2$,

$$f_Y(y) = P_Y[Y = y] = \sum_{x_1=1}^{\infty} P_{X_1, X_2}[X_1 = x_1, X_2 = y - x_1] = \sum_{x_1=1}^{y-1} P_{X_1}[X_1 = x_1] P_{X_2}[X_2 = y - x_1]$$

by independence. Thus

$$\begin{aligned} f_Y(y) &= \sum_{x_1=1}^{y-1} (1 - \theta_1)^{x_1-1} \theta_1 (1 - \theta_2)^{y-x_1-1} \theta_2 \\ &= \frac{\theta_1 \theta_2 (1 - \theta_2)^y}{(1 - \theta_1)(1 - \theta_2)} \sum_{x_1=1}^{y-1} \left(\frac{1 - \theta_1}{1 - \theta_2} \right)^{x_1} \\ &= \frac{\theta_1 \theta_2 (1 - \theta_2)^y}{(1 - \theta_1)(1 - \theta_2)} \left(\frac{1 - \theta_1}{1 - \theta_2} \right) \frac{1 - \left(\frac{1 - \theta_1}{1 - \theta_2} \right)^{y-1}}{1 - \left(\frac{1 - \theta_1}{1 - \theta_2} \right)} \\ &= \theta_1 \theta_2 (1 - \theta_2)^{y-2} \frac{1 - \left(\frac{1 - \theta_1}{1 - \theta_2} \right)^{y-1}}{1 - \left(\frac{1 - \theta_1}{1 - \theta_2} \right)} \\ &= \frac{\theta_1 \theta_2}{\theta_1 - \theta_2} [(1 - \theta_2)^{y-1} - (1 - \theta_1)^{y-1}] \end{aligned}$$

Alternately, using pgfs, we have that

$$G_Y(t) = G_{X_1}(t) G_{X_2}(t) = \frac{\theta_1 t}{1 - t(1 - \theta_1)} \frac{\theta_2 t}{1 - t(1 - \theta_2)}$$

which, on expansion as a power series in t , yields the same summation as above.

3 MARKS

2 If such a transformation exists, and is 1-1, then we must have

$$f_X(x) = f_U(g^{-1}(x)) |J(x)|$$

for inverse transformation g^{-1} , where $|J(x)|$ is the Jacobian

$$|J(x)| = \left| \left\{ \frac{dg^{-1}(t)}{dt} \right\}_{t=x} \right|$$

But, here, $f_U(u) = 1$ for $0 < u < 1$ and zero otherwise. So we merely need to find g^{-1} by solving

$$|J(x)| = \frac{2}{\pi} \frac{1}{\sqrt{1 - x^2}}$$

and hence by a standard result

$$g^{-1}(x) = \frac{2}{\pi} \arcsin(x) \quad \therefore \quad g(x) = \sin\left(\frac{\pi x}{2}\right)$$

which is a 1-1 function for $0 < x < 1$.

3 MARKS

3 By inspection, we have for $(x_1, x_2) \in \mathbb{R}^2$

$$f_{X_1, X_2}(x_1, x_2) = c \exp\{-|x_1|\} |x_1| \exp\left\{-\frac{x_1^2 x_2^2}{2}\right\} = f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1)$$

where

$$f_{X_1}(x_1) = \begin{cases} \frac{1}{2} e^{-|x_1|} & x_1 \in \mathbb{R} \setminus \{0\} \\ 0 & x_1 = 0 \end{cases}$$

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \left(\frac{x_1^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{x_1^2 x_2^2}{2}\right\} & x_2 \in \mathbb{R} \text{ if } x_1 \neq 0 \\ 0 & x_2 \in \mathbb{R} \text{ if } x_1 = 0 \end{cases}$$

Thus $c = 1/\sqrt{8\pi}$.

4 MARKS

4 (a) Random variable $\mathbb{I}_B(X)$ takes values on the set $\{0, 1\}$, with

$$P_{\mathbb{I}_B(X)}[\mathbb{I}_B(X) = 1] = P_X[X \in B] = \theta_B$$

say, so $\mathbb{I}_B(X) \sim \text{Bernoulli}(\theta_B)$, with expectation, from the formula sheet, θ_B

2 MARKS

(b) Let $\mathbb{I}_B(\underline{X})$ be the scalar indicator random variable associated with the event $\underline{X} \in B$. Then from above, we have that

$$\mathbb{E}_{f_{\mathbb{I}_B(\underline{X})}}[\mathbb{I}_B(\underline{X})] = P_{\underline{X}}[\underline{X} \in B]$$

which indicates that we can construct the approximation

$$\hat{\mathbb{E}}_{f_{\mathbb{I}_B(\underline{X})}}[\mathbb{I}_B(\underline{X})] = \frac{1}{N} \sum_{i=1}^N \mathbb{I}_B(\underline{x}_i)$$

where $\underline{x}_1, \dots, \underline{x}_N$ are an independent sample from the specified normal distribution. The following R code implements this:

```
library(MASS)
N<-10000
Sigma<-matrix(c(1,0.2,-0.5,0.2,2.0,-0.1,-0.5,-0.1,2.0),3,3,byrow=T)
set.seed(101)
for(i in 1:5){
  X<-mvrnorm(N,mu=c(0,0,0),Sigma)
  IndX<-X[,1]+X[,3]-(X[,1]^2+X[,2]^2) > 0
  E<-sum(IndX)/N
  print(format(E,nsml=4))
}
```

8 MARKS

and yields the results

```
[1] 0.1513
[1] 0.1433
[1] 0.1475
[1] 0.1529
[1] 0.1484
```

By using a very large N , we can discover that the true value is 0.1468 to four decimal places.