Final Exam MATH 556 : SOLUTIONS

PROBLEM 1.

① Define the mapping $g : (0,\infty) \to (0,1)$

$$g(x) = \frac{px}{q+px}$$

①

$$\frac{px}{q+px} = y \iff px = yq + ypx \iff x(p - yp) = yq$$

$$\iff x = \frac{yq}{p - yp} = \frac{q}{p}\left(\frac{y}{1-y}\right)$$

Hence, $g^{-1} : (0,1) \to (0,\infty)$

① $$y \longmapsto \frac{q}{p}\left(\frac{y}{1-y}\right)$$

and $(g^{-1})'(y) = \dfrac{q}{p}\left(\dfrac{1-y+y}{(1-y)^2}\right) = \dfrac{q}{p}\,\dfrac{1}{(1-y)^2}$

①

Because g is one-to-one (increasing), we have that

① $$f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{q}{p}\,\frac{1}{(1-y)^2}$$

$$= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{p/2} \left(\frac{q}{p}\right)^{\frac{p}{2}-1} \frac{y^{p/2-1}}{(1-y)^{p/2-1}} \times$$

$$\left(1 + \frac{p}{q}\cdot\frac{q}{p}\,\frac{y}{1-y}\right)^{-\frac{p}{2}-\frac{q}{2}} \cdot \frac{q}{p} \cdot \frac{1}{(1-y)^2}$$

① $$= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\, y^{p/2-1}\,(1-y)^{-\frac{p}{2}-1}\,(1-y)^{\frac{p}{2}+\frac{q}{2}}$$

$$= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\, y^{p/2-1}\,(1-y)^{\frac{q}{2}-1} \sim B\left(\frac{p}{2},\frac{q}{2}\right)$$

(b) Define the transformation

$$g : \mathbb{R}\times\mathbb{R}\setminus\{0\} \to \mathbb{R}\times\mathbb{R}\setminus\{0\}$$
$$(x, y) \to \left(\frac{x}{y}, y\right)$$

1
$$g^{-1} : \mathbb{R}\times\mathbb{R}\setminus\{0\} \to \mathbb{R}\times\mathbb{R}\setminus\{0\}$$
$$(u, v) \to (uv, v)$$

Jacobian:

1 $$|J| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

Hence,

1 $$f_{(W,Z)}(w, z) = f_U(w\cdot z)\, f_V(z)\, |z|$$

1 $$f_W(w) = \int_{-\infty}^{\infty} f_U(w\cdot z)\, f_V(z)\, |z|\, dz.$$

(c) $U \sim \chi^2_p$, $V \sim \chi^2_q$ independent.

$$P\left(\frac{U}{V}\cdot\frac{q}{p} \le z\right) = P\left(\frac{U}{V} \le z\cdot\frac{p}{q}\right)$$

Hence $$f_Z(z) \overset{1}{=} f_{\frac{U}{V}}\left(z\cdot\frac{p}{q}\right)\cdot\frac{p}{q}$$

For $t>0$,

$$f_{\frac{U}{V}}(t) = \int_{0}^{\infty} |v| \frac{1}{\Gamma(\frac{p}{2})2^{p/2}} (tv)^{\frac{p}{2}-1} e^{-\frac{tv}{2}} \frac{1}{\Gamma(\frac{q}{2})2^{q/2}} v^{\frac{q}{2}-1} e^{-\frac{v}{2}}\, dv$$

1 $$= \frac{1}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \int_0^\infty v^{\frac{p+q}{2}-1}\, t^{\frac{p}{2}-1} \frac{1}{2^{\frac{p+q}{2}}} e^{-\frac{v}{2}(1+t)}\, dv$$

$$= \frac{t^{\frac{p}{2}-1}}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \cancel{(1+t)} \int_0^\infty 2^{\frac{p+q}{2}-1}\, y^{\frac{p+q}{2}-1}\, (1+t)^{1-\frac{p+q}{2}}\, \frac{1}{2^{\frac{p+q}{2}}}\, e^{-y}\, \frac{2}{(1+t)}$$

$$y = \frac{v}{2}(1+t)$$

$$dy = \frac{dv}{2}(1+t)$$

$$= \frac{t^{\frac{p}{2}-1}}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} (1+t)^{-\frac{p+q}{2}} \int_0^\infty y^{\frac{p+q}{2}-1} e^{-y}\, dy$$

$$\overset{(1)}{=} \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}\, t^{\frac{p}{2}-1} (1+t)^{-\frac{p+q}{2}}$$

$$\Rightarrow \quad f_Z(z) \overset{(1)}{=} \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \left(\frac{p}{q}\right)^{\frac{p}{2}} z^{\frac{p}{2}-1} \left(1+\frac{p}{q}\cdot z\right)^{-\frac{p+q}{2}}, \; z>0$$

(d) $\dfrac{S_1^2}{S_2^2} \sim F_{p,q}$ if

(1) • Samples are independent

• $Y_i \underset{(1)}{\sim} N(\mu_2, \sigma^2)$, $X_i \sim N(\underset{(1)}{\mu_1}, \sigma^2)$

• The variance is the same.

(1)

• $p = m-1$, $q = n-1$

PROBLEM 2

(a) $T = X + Y, \qquad Z = \dfrac{X}{X+Y}$

Define the mapping:

(1) $$g : \mathbb{R}^2 \setminus \{(x,y) : x+y=0\} \longrightarrow \mathbb{R}\setminus\{0\} \times \mathbb{R}$$
$$(x,y) \longrightarrow \left(x+y, \frac{x}{x+y}\right)$$

$$g^{-1}\ (t, z) \longmapsto (tz,\ t - tz)$$

$$|J| = \begin{vmatrix} z & t \\ 1-z & -t \end{vmatrix} = |-zt + t + tz| = |t|$$ (1)

$$f_{(T,Z)}(t,z) \overset{(1)}{=} f_X(tz)\, f_Y(t-tz)\, |t|$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (tz)^{\alpha-1} e^{-tz} (t-tz)^{\beta-1} e^{-t+tz} |t|$$

$$\mathbb{1}(tz>0)\, \mathbb{1}(t-tz>0)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} z^{\alpha-1}(1-z)^{\beta-1} \mathbb{1}(z \in (0,1))$$ (2)

$$\times \frac{1}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1} e^{-t}\, \mathbb{1}(t>0)$$

(b) $$f_Z(z) \overset{(1)}{=} \int_0^\infty f_{(T,Z)}(t,z)\,dt \overset{(1)}{=} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} z^{\alpha-1}(1-z)^{\beta-1} \mathbb{1}(z \in (\ldots$$

$\Rightarrow Z \sim \text{Beta}(\alpha, \beta)$ (1)

By symmetry $\quad W \sim \text{Beta}(\beta, \alpha)$ (1)

(c) Because $f_{(T,Z)}(t,z) \stackrel{1}{=} f_T(t)\, f_Z(t)$, $\quad T \perp Z$

and hence $\quad \mathrm{cor}(T, Z) \stackrel{1}{=} 0$.

Furthermore,

$$\mathrm{cor}(Z, W) = \mathrm{cor}(Z, 1 - Z)$$

$$1 \qquad = \mathrm{cor}(Z, 1) - \mathrm{cor}(Z, Z)$$

$$= -\,\mathrm{var}(Z) = -\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \neq 0$$

1

$\Rightarrow$ Z and W are not independent.

PROBLEM 3

(a) $$f(x \mid \alpha) = \alpha \left(\frac{1}{1+x}\right)^{\alpha+1}, \quad x > 0$$

$$= \alpha\, e^{-(\alpha+1)\log(1+x)}\, 1(x > 0)$$

$$1 \qquad = 1(x>0)\, \frac{1}{1+x} \cdot \alpha \cdot e^{-\alpha \cdot \log(1+x)}.$$

$$h(x) = 1(x>0)\, \frac{1}{1+x}$$

$$c(\alpha) = \alpha \qquad 1$$

$$w(\alpha) = +\alpha$$

$$t(x) = -\log(1+x).$$

This is an exponential family already in its natural form. 1

Natural parameter space: (1)

$$\left\{ \alpha : \int_0^\infty \left(\frac{1}{x+1} \right)^{\alpha+1} dx < \infty \right\} = (0, \infty).$$

(b) By the formulas used in class,

$$E(-\log(1+X)) \overset{1}{=} -\frac{\partial}{\partial \alpha} \log c(\alpha)$$

$$E(\log(1+X)) \overset{1}{=} \frac{\partial}{\partial \alpha} \log \alpha = \frac{1}{\alpha}.$$

$$var(\log(1+X)) \overset{1}{=} -\frac{\partial^2}{\partial^2 \alpha} \log \alpha = \frac{1}{\alpha^2}$$

(c) Suppose that $g(x)$ has MGF $M(x)$.

$$M(t) = \int e^{tx} g(x)\, dx \iff 1 = \int e^{tX - K(t)} g(x)\, dx$$

where $K(t) \overset{1}{=} \log M(t)$.

$$g(x|t) \overset{2}{=} \underbrace{g(x)}_{h(x)} \cdot e^{tx} \cdot \underbrace{e^{-K(t)}}_{c(t)}$$

(d) No, because a Pareto distr. does not have an MGF:

(2) $$M(t) = \int_0^\infty e^{tx} \left(\frac{1}{1+X} \right)^{\alpha+1} \cdot \alpha \, dx = \infty \quad \forall\, t > 0.$$

(e) $\mathcal{U}(0, \theta)$:

(4) $$f(x|\theta) = \frac{1}{\theta} \underbrace{1(x \in (0, \theta))}_{\text{this is not of the form } h(x) \cdot e^{\Sigma w(\theta) t(x)}}$$

PROBLEM 4

$X_i \mid P_i \sim \text{Bernoulli}(P_i)$

$P_i \sim \text{Beta}(\alpha, \beta)$

$Y = \sum_{i=1}^{N} X_i$, $N \sim \text{Poisson}(N)$.

(a) MGF of Y is :

$$M_Y(t) = E e^{tY} \overset{1}{=} E\left(E\left(e^{t \sum_{i=1}^{n} X_i} \mid N\right)\right).$$

$$M_{X_i}(t) = E e^{tX_i} = E\left(E\left(e^{tX_i} \mid P_i\right)\right)$$

$$= E(1 - P_i + P_i e^t)$$

$$= 1 - E(P_i) + e^t E(P_i)$$

$$\overset{2}{=} 1 - \frac{\alpha}{\alpha+\beta} + e^t \frac{\alpha}{\alpha+\beta}.$$

This means that $X_i \sim \text{Bernoulli}\left(\frac{\alpha}{\alpha+\beta}\right)$.

$$M_Y(t) \overset{1}{=} E\left(\{M_{X_i}(t)\}^N\right) = M_Y(\log M_{X_i}(t))$$

$$= \exp(\lambda(M_{X_i}(t) - 1))$$

$$= e^{\lambda\left(1 - \frac{\alpha}{\alpha+\beta} + e^t \frac{\alpha}{\alpha+\beta} - 1\right)}$$

$$= e^{\lambda \cdot \frac{\alpha}{\alpha+\beta}(e^t - 1)}$$

$$\overset{1}{\Rightarrow} Y \sim \text{Poisson}\left(\frac{\lambda\alpha}{\alpha+\beta}\right).$$

(2) $E(Y) = \dfrac{\lambda\alpha}{\alpha+\beta} = E(E(Y|N))$

$$= EN\,EX = EN\,E(E(X|P_i))$$

2

$$= \frac{\lambda\alpha}{\alpha+\beta}.$$

$$\operatorname{var}(Y) = E(\operatorname{var}(Y|N)) + \operatorname{var}(E(Y|N))$$

$$= EN\cdot \operatorname{var} X + \operatorname{var} N\cdot (EX)^2$$

2

$$= \lambda\cdot\frac{\alpha}{\alpha+\beta}\frac{\beta}{\alpha+\beta} + \lambda\cdot\left[\frac{\alpha}{\alpha+\beta}\right]^2$$

$$= \lambda\cdot\frac{\alpha}{\alpha+\beta}\left(\frac{\beta+\alpha}{\alpha+\beta}\right) = \frac{\lambda\alpha}{\alpha+\beta}.$$

(5) (1) $\operatorname{cov}(X, Y - E(Y|X))$

2

$$= E(XY - XE(Y|X)) - EX\,E(Y - E(Y|X))$$

$$= EXY - E(X\,E(Y|X)) - EX\,EY + EX\,E(E(Y|X))$$

1 1

$$= EXY - EXY - EX\,EY + EX\,EY = 0.$$

1

(2) $\operatorname{var}(Y - E(Y|X)) = E(\operatorname{var}(Y|X))$

LHS: $\overset{1}{=} E((Y - E(Y|X))^2) - EY + \underbrace{E(E(Y|X))}_{EY\ 1}$

$$\cancel{= E(Y^2 - 2Y E(Y|X) + \{E(Y|X)\}^2)}$$

$$\overset{1}{=} E(E(\{Y - E(Y|X)\}^2|X)) \overset{1}{=} E(\operatorname{var}(Y|X))$$

PROBLEM 5

First, set $S_n = X_1 + \cdots + X_n$.

$$M_{S_n}(t) = \{M_{X_1}(t)\}^n \overset{(2)}{=} \left(\frac{p}{1-e^t(1-p)}\right)^{rn}$$

$\Rightarrow S_n \overset{(1)}{\sim} \text{NegBin}(p, r\cdot n)$.

$\Rightarrow \frac{1}{n} S_n$ has support $\overset{(1)}{\left\{0, \frac{1}{n}, \frac{2}{n}, \dots\right\}}$

and $P\left(\frac{1}{n} S_n = \frac{k}{n}\right) = P(S_n = k) \overset{(1)}{=} \text{NegBin}(k; k, r\cdot n)$

(b) Jensen's inequality: $E g(X) \geq g(EX)$.

g is convex means that at any x_0,

$$g(x) \overset{(2)}{\geq} g(x_0) + g'_+(x_0)(X - X_0)$$

In particular, for $X_0 \overset{(1)}{=} EX$,

$$g(x) \geq g(E(X)) + g'_+(E(X))(X - EX)$$

$$\Rightarrow \quad E(g(X)) \overset{(2)}{\geq} g(EX) + \underbrace{g'_+(E(X))(EX - EX)}_{0}$$

(c) We have that $E(\overline{X}_n) \overset{(1)}{=} EX = \frac{(1-p)(r)}{p}$

$$g(x) = \frac{r}{x+r}$$

$$g'(x) = \frac{-r}{(x+r)^2} \quad ; \quad g''(x) = \frac{2r}{(x+r)^3} > 0 \text{ on } (0, \infty)$$

so g is convex, and (*)

$$E(T_n) \geq g(E\bar{X}_n) = \frac{r}{r\frac{(1-p)}{p} + r} = \frac{1}{\frac{1}{p}} = p. \quad (1)$$

$\bar{X}_n \xrightarrow{P} \frac{r(1-p)}{p}$ by WLLN and hence $T_n \xrightarrow{P} p$ by the CLT.

(d) $X \sim \text{NegBin}(r, p)$

$$X \overset{d}{=} Y_1 + \cdots + Y_r, \qquad Y_i \sim \text{NegBin}(1, p) \quad (2)$$

by part (a). Now, if r is large,

$$\sqrt{r}\,\frac{\frac{Y_1 + \cdots + Y_r}{r} - \frac{r\cancel{p}(1-p)}{p}}{\sqrt{\frac{r(1-p)}{p^2}}} \overset{(1)}{\approx} N(0,1).$$

$$P(X \leq x) = P(Y_1 + \cdots + Y_r \leq x)$$

$$= P\left(\bar{Y}_r \leq \frac{x}{r}\right) = P\left(\sqrt{r}\,\frac{\bar{Y}_r - \frac{r(1-p)}{p}}{\sqrt{}} \leq \sqrt{r}\,\frac{\frac{x}{r} - \frac{r(1-p)}{p}}{\sqrt{}}\right) \quad (2)$$

$$\approx \Phi\left(\sqrt{r}\cdot\frac{\frac{x}{r} - \frac{r(1-p)}{p}}{\sqrt{\frac{r(1-p)}{p^2}}}\right)$$

PROBLEM 6

a) $P(M_n \le x) \overset{1}{=} P(X_1 \le x, \dots, X_n \le x)$

$$\overset{2}{=} \left\{ 1 - (1-x)^{\alpha} \right\}^n, \quad x \in (0,1).$$

(b) To show: $\forall \varepsilon > 0$,

$$\lim_{n \to \infty} P(|Y_n - a| > \varepsilon) \overset{1}{=} 0.$$

$$P(|Y_n - a| > \varepsilon) \overset{1}{=} P(Y_n < a - \varepsilon) + P(Y_n > a + \varepsilon)$$

$$\overset{1}{\le} \underset{\overset{\downarrow}{0}}{P(Y_n \le a - \varepsilon)} + 1 - \underset{\overset{\downarrow}{1}}{P(Y_n \le a + \varepsilon)}$$

2

$$\to 0 \quad \text{as } n \to \infty.$$

(c)

$$P(M_n \le x) \longrightarrow \begin{cases} 0, & x < 0 \\ 0, & x < 1 \\ 1, & x \ge 1. \end{cases}$$

2

$M_n \rightsquigarrow 1$ and by (b), $M_n \xrightarrow{P} 1$.

2

(d)

$$P\left(n^{\frac{1}{\alpha}}(M_n - 1) \le x\right)$$

$$= P\left(M_n - 1 \le x \cdot n^{-\frac{1}{\alpha}}\right) = P\left(M_n \le x \cdot n^{-\frac{1}{\alpha}} + 1\right)$$

$$\overset{*}{=} \left(1 - \left(1 - x \cdot n^{-\frac{1}{\alpha}} + 1\right)^{\alpha}\right)^{n}, \quad x \in \left(-n^{\frac{1}{\alpha}}, 0\right) \quad (1)$$

(2)

$$= \left(1 - \frac{(-x)^{\alpha}}{n}\right)^{n} \longrightarrow e^{-(-x)^{\alpha}} \quad (1)$$

* if $x \cdot n^{-\frac{1}{\alpha}} + 1 \in (0, 1)$

$$0 < x \cdot n^{-\frac{1}{\alpha}} + 1 \iff -n^{\frac{1}{\alpha}} < x$$

$$x \cdot n^{-\frac{1}{\alpha}} + 1 < 1$$

$$x < 0.$$

For all $x < 0$, $P\left(n^{\frac{1}{\alpha}}(M_n - 1) \le x\right) \to e^{-(-x)^{\alpha}}$ (1)

$x \ge 0$: $P(\qquad) = 1 \to 1.$