MATH 556 – MID-TERM EXAMINATION

Marks can be obtained by answering all questions. All questions carry equal marks.

The total mark available is 60, but rescaling of the final mark may occur.

Show your working in all cases. You may quote results from the formula sheet or from lecture notes if you make explicit the result you are quoting.

1. Identify which of the following functions are legitimate specifications for pmfs or pdfs (denoted f) or cdfs (denoted F for a scalar random variable), giving reasons as to your conclusions. Note that $\mathbb{1}_A(x)$ is the indicator function for set A that takes the value 1 if $x \in A$ and zero otherwise.

   (a)
$$f(x) = \mathbb{1}_A(x) \qquad A \equiv \{1, 3, 5\}.$$
3 MARKS

   (b) For some constant c,
$$f(x) = c\mathbb{1}_A(x)\frac{e^x}{x!} \qquad A \equiv \{1, 2, \ldots, 100\}.$$
3 MARKS

   (c) For some constant c,
$$f(x) = c\{\Phi(x)\}^3\phi(x) \qquad x \in \mathbb{R}$$
where $\Phi(x)$ and $\phi(x)$ are the cdf and pdf of the standard Normal distribution, respectively.
3 MARKS

   (d) For $\lambda > 0$,
$$F(x) = \frac{1}{2}\mathbb{1}_{(0,1)}(x)x + \frac{1}{2}\mathbb{1}_{[1,\infty)}(x)(1 - e^{-\lambda(x-1)}).$$
3 MARKS

   (e) For some constants c_1, c_2,
$$F(x) = \begin{cases} 0 & x < 0 \\ c_1 x^2 & 0 \le x < 1 \\ c_2\sqrt{x} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}.$$
3 MARKS

2. Suppose that X and Y have a joint probability distribution that is specified using marginal and conditional distributions as follows:
$$X \sim Bernoulli(\theta)$$
$$Y|X = x \sim Normal(2x - 1, 1)$$
for some θ, $0 < \theta < 1$.

   (a) Find the marginal pdf for the continuous random variable Y, and find the expectation and variance of Y.
9 MARKS

   (b) Suppose that X_1 and X_2 are independent, and have the same distribution as X. Find the pmf for the random variable
$$Z = X_1 + 2X_2.$$
Hint: identify the support of f_Z.
6 MARKS

3. (a) Suppose X is a discrete random variable where

$$P_X[X = -1] = \frac{1}{4} \qquad P_X[X = 0] = \frac{1}{2} \qquad P_X[X = 1] = \frac{1}{4}.$$

(i) Write down the quantile function for X, $Q_X(p)$ for $0 < p < 1$. 5 MARKS

(ii) Find the pmf for random variable Y defined by $Y = X^2$. 2 MARKS

(iii) Find $\mathbb{E}_X[X^3]$. 3 MARKS

(b) Compute the Kullback-Leibler (KL) divergence between the pmfs f_0 and f_1, $KL(f_0, f_1)$, if f_0 is the $Poisson(\lambda_0)$ pmf, and f_1 is the $Poisson(\lambda_1)$ pmf.

Note that the KL divergence

$$KL(f_0, f_1) = \mathbb{E}_{f_0}\left[\log \frac{f_0(X)}{f_1(X)}\right]$$

is computed using summation rather than integration for discrete pmfs.

5 MARKS

4. Suppose that X_1 and X_2 are independent $Exponential(1)$ random variables.

(a) Compute

$$P_{X_1,X_2}\left[\frac{X_1}{X_2} > 1\right].$$

3 MARKS

(b) Find the pdf of the random variable $Y = X_1 - X_2$.

Hint: from first principles

$$P_Y[Y \leq y] = P_{X_1,X_2}[X_1 - X_2 \leq y] = \iint_{A_y} f_{X_1,X_2}(x_1, x_2)\, dx_1 dx_2$$

for $y \in \mathbb{R}$, where

$$A_y = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 \leq y\right\}.$$

Consider the cases $y < 0$ and $y > 0$ separately. 12 MARKS