Asymptotic Approximations And The Delta Method

Asymptotic Approximations

To approximate the distribution of elements in sequence of random variables $\{X_n\}$ for large n, we attempt to find sequences of constants $\{a_n\}$ and $\{b_n\}$ such that

$$Z_n = a_n X_n + b_n \xrightarrow{d} Z$$

where Z has some distribution characterized by cdf F_Z. Then, for large n, $F_{Z_n}(z) \doteq F_Z(z)$, so

$$\begin{aligned} F_{X_n}(x) &= P[X_n \leq x] = P[a_n X_n + b_n \leq a_n x + b_n] \\ &= F_{Z_n}(a_n x + b_n) \\ &\doteq F_Z(a_n x + b_n). \end{aligned}$$

Asymptotic Approximations (cont.)

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. such that $X_i \sim Exp(1)$, and let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$. Then for $y > 0$,

$$F_{Y_n}(y) = \{F_X(y)\}^n = \{1 - e^{-y}\}^n \longrightarrow 0$$

and there is no limiting distribution. However, if we take $a_n = 1$ and $b_n = -\log n$, and set $Z_n = a_n Y_n + b_n$, then as $n \longrightarrow \infty$,

$$\begin{aligned} F_{Z_n}(z) &= \mathrm{P}[Z_n \leq z] = \mathrm{P}[Y_n \leq z + \log n] = \{1 - e^{-z-\log n}\}^n \\ &\longrightarrow \exp\{-e^{-z}\} = F_Z(z), \\ \therefore \quad F_{Y_n}(y) &= P[Y_n \leq y] = P[Z_n \leq y - \log n] \doteq F_Z(y - \log n) \\ &= \exp\{-e^{-y+\log n}\} = \exp\{-ne^{-y}\} \end{aligned}$$

and by differentiating, for $y > 0$

$$f_{Y_n}(y) \doteq ne^{-y} \exp\{-ne^{-y}\}.$$

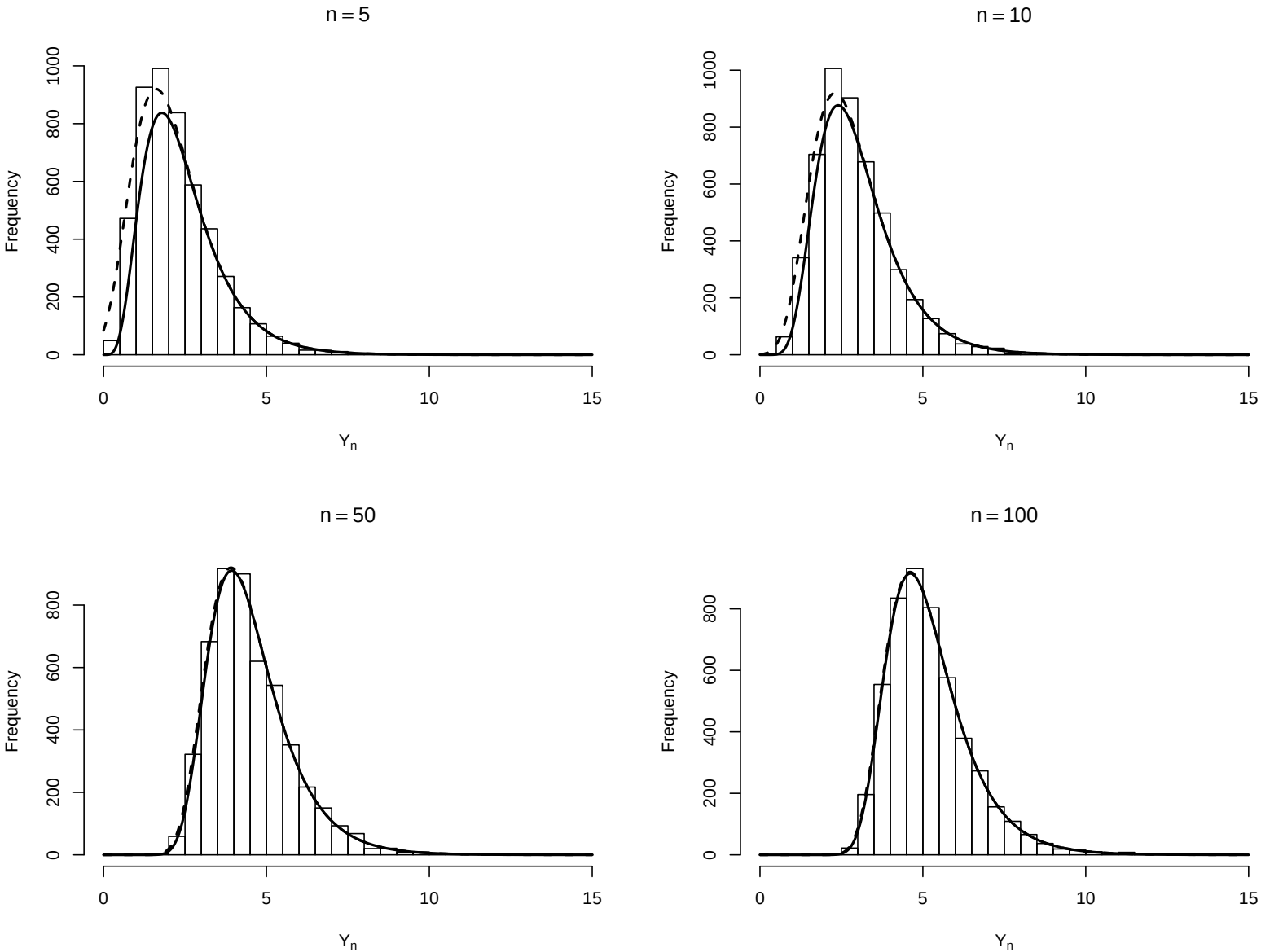
Asymptotic Approximations (cont.)

This can be compared with the exact version, for $y > 0$

$$f_{Y_n}(y) = ne^{-y}(1 - e^{-y})^{n-1}.$$

The figure below compares the approximations for $n = 50, 100, 500, 1000$. Solid lines use the exact formula, dotted lines use the approximation, histograms are 5000 simulated values.

Asymptotic Approximations



Stochastic Order Notation

A sequence of random variables $\{X_n\}$ is **asymptotically normally distributed** as $n \longrightarrow \infty$ if there exist sequences of real constants $\{\mu_n\}$ and $\{\sigma_n\}$ (with $\sigma_n > 0$) such that

$$\frac{X_n - \mu_n}{\sigma_n} \xrightarrow{d} Z \sim \mathcal{N}(0, 1).$$

The notation

$$X_n \dot{\sim} \mathcal{N}(\mu_n, \sigma_n^2) \qquad \text{or} \qquad X_n \sim \mathcal{AN}(\mu_n, \sigma_n^2)$$

as $n \longrightarrow \infty$ is commonly used.

Stochastic Order Notation (cont.)

- For random variable Z, we write $Z = O_p(1)$ if for all $\epsilon > 0$, there exists $M < \infty$ such that

$$P[|Z| \geq M] \leq \epsilon.$$

- For sequence $\{Z_n\}$, write $Z_n = O_p(1)$ if for all n, $P[|Z_n| \geq M] \leq \epsilon$, and write $Z_n = O_p(S_n)$ for sequence of random variables $\{S_n\}$ if

$$\frac{|Z_n|}{|S_n|} = O_p(1).$$

  Note that this includes the case where S_n is a sequence of reals, rather than random variables.

Stochastic Order Notation (cont.)

Finally, write $Z_n = o_p(1)$ if $Z_n \xrightarrow{p} 0$, and $Z_n = o_p(S_n)$ if

$$\frac{|Z_n|}{|S_n|} = o_p(1).$$

Note that $O_p(1)o_p(1) = o_p(1)$ and $O_p(1) + o_p(1) = O_p(1)$.

Transforms

Suppose $\{X_n\}$ are a sequence of rvs, and that for real sequence $\{a_n\}$ with $a_n \longrightarrow \infty$ as $n \longrightarrow \infty$,

(i) for real constant x_0 and random variable V,

$$a_n(X_n - x_0) \xrightarrow{d} V;$$

(ii) real function g is differentiable at x_0, with derivative $\dot{g}$.

Then

$$a_n(g(X_n) - g(x_0)) \xrightarrow{d} \dot{g}(x_0)\, V$$

Transforms (cont.)

Note first that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| \leq \delta \quad \Longrightarrow \quad |g(x) - g(x_0) - \dot{g}(x_0)(x - x_0)| \leq \epsilon |x - x_0|$$

Now, from (i) we have

$$a_n(X_n - x_0) = O_p(1) \qquad \Longrightarrow \qquad X_n - x_0 = O_p(a_n^{-1}) = o_p(1)$$

as $a_n \longrightarrow \infty$. Therefore, by definition, for every $\delta > 0$,
$P[|X_n - x_0| \leq \delta] \longrightarrow 1$, and therefore from above, for every $\epsilon > 0$,

$$P[|g(X_n) - g(x_0) - \dot{g}(x_0)(X_n - x_0)| \leq \epsilon |X_n - x_0|] \longrightarrow 1.$$

Transforms (cont.)

Hence

$$a_n(g(X_n) - g(x_0) - \dot{g}(x_0)(X_n - x_0)) = o_p(a_n(X_n - x_0)) = o_p(1)$$

Therefore

$$a_n(g(X_n) - g(x_0)) = \dot{g}(x_0)\{a_n(X_n - x_0)\} + o_p(1)$$

and hence

$$a_n(g(X_n) - g(x_0)) \xrightarrow{d} \dot{g}(x_0)\, V.$$

The Delta Method

THEOREM (The Delta Method)
Consider sequence of random variables $\{X_n\}$ such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X.$$

Suppose that $g(.)$ is a function such that first derivative $\dot{g}(.)$ is continuous in a neighbourhood of μ, with $\dot{g}(\mu) \neq 0$. Then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X.$$

In particular, if

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2).$$

then

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \dot{g}(\mu)X \sim \mathcal{N}(0, \{\dot{g}(\mu)\}^2\sigma^2).$$

The Delta Method (cont.)

Using the Lemma above, with $a_n = \sqrt{n}, x_0 = \mu, V = X$, we have that

$$\sqrt{n}(g(X_n) - g(\mu)) = \dot{g}(\mu)\sqrt{n}(X_n - \mu) \xrightarrow{d} \dot{g}(\mu)X$$

and if $X \sim \mathcal{N}(0, \sigma^2)$, it follows from the properties of the Normal distribution that

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{d} \mathcal{N}(0, \{\dot{g}(\mu)\}^2\sigma^2).$$

Note that this method does not give a useful result if $\dot{g}(\mu) = 0$.

The Delta Method (cont.)

Multivariate Version: Consider a sequence of random vectors $\{\mathbf{X}_n\}$ such that

$$\sqrt{n}(\mathbf{X}_n - \boldsymbol{\mu}) \xrightarrow{d} \mathbf{X}.$$

and $\mathbf{g} : \mathbb{R}^k \longrightarrow \mathbb{R}^d$ is a vector-valued function with first derivative matrix $\dot{\mathbf{g}}(.)$ which is continuous in a neighbourhood of $\boldsymbol{\mu}$, with $\dot{g}(\boldsymbol{\mu}) \neq \mathbf{0}$. Note that $\mathbf{g}$ can be considered as a $d \times 1$ vector of scalar functions.

$$\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_d(\mathbf{x}))^{\mathsf{T}}.$$

Note that $\dot{\mathbf{g}}(\mathbf{x})$ is a $(d \times k)$ matrix with $(i,j)th$ element

$$\frac{\partial g_i(\mathbf{x})}{\partial x_j}$$

Under these assumptions, in general

$$\sqrt{n}(\mathbf{g}(\mathbf{X}_n) - \mathbf{g}(\boldsymbol{\mu})) \xrightarrow{d} \dot{\mathbf{g}}(\boldsymbol{\mu})\mathbf{X}.$$

The Delta Method (cont.)

and in particular, if

$$\sqrt{n}(\mathbf{X}_n - \boldsymbol{\mu}) \xrightarrow{d} \mathbf{X} \sim \mathcal{N}(0, \Sigma).$$

where Σ is a positive definite, symmetric $k \times k$ matrix, then

$$\sqrt{n}(\mathbf{g}(\mathbf{X}_n) - \mathbf{g}(\boldsymbol{\mu})) \xrightarrow{d} \dot{\mathbf{g}}(\boldsymbol{\mu})X \sim \mathcal{N}\left(0, \dot{\mathbf{g}}(\boldsymbol{\mu})\Sigma\dot{\mathbf{g}}(\boldsymbol{\mu})^{\mathsf{T}}\right).$$

The Delta Method (cont.)

THEOREM (The Second Order Delta Method: Normal case) Consider sequence of random variables $\{X_n\}$ such that

$$\sqrt{n}(X_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Suppose that $g(.)$ is a function such that first derivative $\dot{g}(.)$ is continuous in a neighbourhood of μ, with $\dot{g}(\mu) = 0$, but second derivative exists at μ with $\ddot{g}(\mu) \neq 0$. Then

$$n(g(X_n) - g(\mu)) \xrightarrow{d} \sigma^2 \frac{\ddot{g}(\mu)}{2} X$$

where $X \sim \chi^2_1$.

Proof.

The Delta Method (cont.)

Uses a second order Taylor approximation; informally

$$g(X_n) = g(\mu) + \dot{g}(\mu)(X_n - \mu) + \frac{\ddot{g}(\mu)}{2}(X_n - \mu)^2 + o_p(1)$$

thus, as $\dot{g}(\mu) = 0$,

$$g(X_n) - g(\mu) = \frac{\ddot{g}(\mu)}{2}(X_n - \mu)^2 + o_p(1)$$

and thus

$$n(g(X_n) - g(\mu)) = \frac{\ddot{g}(\mu)}{2}\{\sqrt{n}(X_n - \mu)\}^2 \xrightarrow{d} \sigma^2 \frac{\ddot{g}(\mu)}{2} Z^2$$

where $Z^2 \sim \chi^2_1$. ∎

Examples

1. Under the conditions of the Central Limit Theorem, for random variables $X_1, \ldots, X_n$ and their sample mean random variable $\overline{X}_n$

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2).$$

Consider $g(x) = x^2$, so that $\dot{g}(x) = 2x$, and hence, if $\mu \neq 0$,

$$\sqrt{n}({\overline{X}_n}^2 - \mu^2) \xrightarrow{d} X \sim \mathcal{N}(0, 4\mu^2\sigma^2)$$

and

$${\overline{X}_n}^2 \sim \mathcal{AN}(\mu^2, 4\mu^2\sigma^2/n)$$

Examples (cont.)

If $\mu = 0$, we proceed by a different route to compute the approximate distribution of $\overline{X}_n{}^2$; note that, if $\mu = 0$,

$$\sqrt{n}\overline{X}_n \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2)$$

so therefore

$$n\overline{X}_n{}^2 = (\sqrt{n}\overline{X}_n)^2 \xrightarrow{d} X^2 \sim \text{Gamma}(1/2, 1/(2\sigma^2))$$

by elementary transformation results. Hence, for large n,

$$\overline{X}_n{}^2 \dot{\sim} \text{Gamma}(1/2, n/(2\sigma^2))$$

Examples (cont.)

2. Again under the conditions of the CLT, consider the distribution of $1/\overline{X}_n$. In this case, we have a function $g(x) = 1/x$, so $\dot{g}(x) = -1/x^2$, and if $\mu \neq 0$, the Delta method gives

$$\sqrt{n}(1/\overline{X}_n - 1/\mu) \xrightarrow{d} X \sim \mathcal{N}(0, \sigma^2/\mu^4)$$

or,

$$\frac{1}{\overline{X}_n} \sim \mathcal{AN}(1/\mu, n^{-1}\sigma^2/\mu^4).$$

The Joint Distribution Of Sample Quantiles

RESULT 1: If $Y_1, Y_2, \ldots, Y_{n+1} \sim$ Exponential (1) are independent random variables, and $S_1, S_2, \ldots, S_{n+1}$ are defined by

$$S_k = \sum_{j=1}^{k} Y_j \qquad k = 1, 2, \ldots, n+1$$

then the random variables

$$\left[\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}} \right]$$

given that $S_{n+1} = s$, say, have the same distribution as the order statistics from a random sample of size n from the Uniform distribution on $(0, 1)$.

The Joint Distribution Of Sample Quantiles (cont.)

Proof: Let the Y_js be defined as above. Then the joint density for the Y_js is given by

$$\exp\left\{-\sum_{j=1}^{n+1} y_j\right\} \qquad y_1, y_2, \ldots, y_{n+1} > 0.$$

Now

$$\left.\begin{array}{rcl} S_1 &=& Y_1 \\ S_2 &=& Y_1 + Y_2 \\ S_3 &=& Y_1 + Y_2 + Y_3 \\ \vdots & & \vdots \\ S_n &=& \sum_{j=1}^{n} Y_j \\ S_{n+1} &=& \sum_{j=1}^{n+1} Y_j \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{rcl} Y_1 &=& S_1 \\ Y_2 &=& S_2 - S_1 \\ Y_3 &=& S_3 - S_2 \\ \vdots & & \vdots \\ Y_n &=& S_n - S_{n-1} \\ Y_{n+1} &=& S_{n+1} - S_n \end{array}\right.$$

The Joint Distribution Of Sample Quantiles (cont.)

and so the Jacobian of the transformation from

$$(Y_1, \ldots, Y_{n+1}) \longrightarrow (S_1, \ldots, S_{n+1})$$

is 1, and hence the joint density for $(S_1, \ldots, S_{n+1})$ is given by

$$\exp\{-s_{n+1}\} \qquad 0 < s_1 < s_2 < \ldots < s_{n+1} < \infty.$$

The marginal distribution for S_{n+1} is Gamma $(n+1, 1)$ and thus the conditional distribution of $(S_1, \ldots, S_n)$ given $S_{n+1} = s$ is

$$\frac{\exp\{-s\}}{\frac{1}{\Gamma(n+1)} s^n \exp\{-s\}} = \frac{n!}{s^n} \qquad 0 < s_1 < s_2 < \ldots < s < \infty.$$

The Joint Distribution Of Sample Quantiles (cont.)

Finally, conditional on $S_{n+1} = s$, define the joint transformation

$$V_j = \frac{S_j}{s} \Longleftrightarrow S_j = sV_j \qquad j = 1, 2, \ldots, n$$

which has Jacobian s^n. Then, conditional on $S_{n+1} = s$, $(V_1, \ldots, V_n)$ have joint pdf equal to $n!$ for $0 < v_1 < v_2 < \ldots < v_n < 1$.

Finally, if $U_1, \ldots, U_n$ are independent random variables each having a Uniform distribution on $(0, 1)$, then $(U_1, \ldots, U_n)$ have joint pdf equal to 1 on the unit hypercube in n dimensions, and thus the corresponding order statistics $U_{(1)}, \ldots, U_{(n)}$ also have joint pdf equal to

$$n! \qquad 0 < u_1 < u_2 < \ldots < u_n < 1.$$

The Joint Distribution Of Sample Quantiles (cont.)

RESULT 2: Let the S_k be defined as in Result 1. Then

$$\sqrt{k}\left(\frac{S_k}{k} - 1\right) \xrightarrow{d} \mathcal{N}(0,1) \text{ as } k \longrightarrow \infty$$

Proof: We have that S_k is the sum of k independent and identically distributed Exponential(1) random variables, $Y_1, \ldots, Y_k$, so that $\mathbb{E}\left[Y_j\right] = \mathbb{V}\text{ar}\left[Y_j\right] = 1$. Thus result follows via the Central Limit Theorem.

RESULT 3: Let the S_k be defined as in Result 1. Then, if k_{1n} is a sequence of integers such that

$$k_{1n} \longrightarrow \infty \qquad \text{while} \qquad \frac{k_{1n}}{n} \longrightarrow p_1$$

The Joint Distribution Of Sample Quantiles (cont.)

for some p_1 with $0 < p_1 < 1$, it follows that

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right) \xrightarrow{d} \mathcal{N}(0, p_1) \text{ as } n \longrightarrow \infty$$

Proof: We have

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right) = \sqrt{\frac{k_{1n}}{n+1}} \times \sqrt{k_{1n}}\left(\frac{S_{k_{1n}}}{k_{1n}} - 1\right) \xrightarrow{d} \sqrt{p_1} \times \mathcal{N}(0,1) \equiv \mathcal{N}(0, p_1)$$

as $n \longrightarrow \infty$ and $k_{1n} \longrightarrow \infty$.

The Joint Distribution Of Sample Quantiles (cont.)

Corollary: Using the same approach, if

$$\frac{k_{1n}}{n} \longrightarrow p_1 \qquad \text{and} \qquad \frac{k_{2n}}{n} \longrightarrow p_2$$

for $0 < p_1 < p_2 < 1$, then if $D_n = \sum_{j=k_{1n}+1}^{k_{2n}} Y_j$,

$$\sqrt{n+1}\left(\frac{(S_{k_{2n}} - S_{k_{1n}})}{n+1} - \frac{k_{2n}-k_{1n}}{n+1}\right) = \sqrt{\frac{k_{2n}-k_{1n}}{n+1}}\sqrt{k_{2n}-k_{1n}}\left(\frac{D_n}{k_{2n}-k_{1n}} - 1\right)$$

$$\xrightarrow{d} \sqrt{p_2 - p_1} \times \mathcal{N}(0,1) \equiv \mathcal{N}(0, p_2 - p_1).$$

Similarly

$$\sqrt{n+1}\left(\frac{1}{n+1}(S_{n+1} - S_{k_{2n}}) - \frac{n+1-k_{2n}}{n+1}\right) \xrightarrow{d} \mathcal{N}(0, 1-p_2)$$

The Joint Distribution Of Sample Quantiles (cont.)

where the limiting variables in the three cases are independent, as

$$
\begin{aligned}
S_{k_{1n}} &= \sum_{j=1}^{k_{1n}} Y_j \\
\left(S_{k_{2n}} - S_{k_{1n}}\right) &= \sum_{j=k_{1n}+1}^{k_{2n}} Y_j \\
\left(S_{n+1} - S_{k_{2n}}\right) &= \sum_{j=k_{2n}+1}^{n+1} Y_j
\end{aligned}
$$

are independent.

The Joint Distribution Of Sample Quantiles (cont.)

RESULT 4: Let

$$Z_1 = \frac{S_{k_{1n}}}{n+1} \qquad Z_2 = \frac{\left(S_{k_{2n}} - S_{k_{1n}}\right)}{n+1} \qquad Z_3 = \frac{\left(S_{n+1} - S_{k_{2n}}\right)}{n+1}$$

and suppose that

$$\sqrt{n}\left(\frac{k_{1n}}{n} - p_1\right) \longrightarrow 0 \qquad \text{and} \qquad \sqrt{n}\left(\frac{k_{2n}}{n} - p_2\right) \longrightarrow 0$$

as $n \longrightarrow \infty$. Then

$$\sqrt{n+1}\left(\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 - p_1 \\ 1 - p_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

as $n \longrightarrow \infty$, where $\Sigma = \text{diag}\,(p_1, p_2 - p_1, 1 - p_2)$.

The Joint Distribution Of Sample Quantiles (cont.)

Proof: We have, as $n \longrightarrow \infty$,

$$\sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - p_1\right) - \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right) = \sqrt{n+1}\left(\frac{k_{1n}}{n+1} - p_1\right) \longrightarrow 0$$

$$\therefore \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - p_1\right) \qquad \text{and} \qquad \sqrt{n+1}\left(\frac{S_{k_{1n}}}{n+1} - \frac{k_{1n}}{n+1}\right)$$

have the same asymptotic distribution, and thus the result follows from Result 3. The proof is similar for the other two terms. Independence (that is, the diagonal nature of Σ) follows from the independence of $S_{k_{1n}}$, $(S_{k_{2n}} - S_{k_{1n}})$, and $(S_{n+1} - S_{k_{2n}})$.

The Joint Distribution Of Sample Quantiles (cont.)

RESULT 5: If $U_{(1)}, \ldots, U_{(n)}$ are the order statistics from a random sample of size n from a Uniform $(0, 1)$ distribution, and if $n \longrightarrow \infty$, $k_{1n} \longrightarrow \infty$ and $k_{2n} \longrightarrow \infty$ in such a way that

$$\sqrt{n}\left(\frac{k_{1n}}{n} - p_1\right) \longrightarrow 0 \qquad \text{and} \qquad \sqrt{n}\left(\frac{k_{2n}}{n} - p_2\right) \longrightarrow 0$$

for $0 < p_1 < p_2 < 1$, then

$$\sqrt{n}\left(\begin{pmatrix} U_{(k_{1n})} \\ U_{(k_{2n})} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{bmatrix}\right).$$

The Joint Distribution Of Sample Quantiles (cont.)

Proof: Define

$$\mathbf{g}(x_1, x_2, x_3) = \frac{1}{x_1 + x_2 + x_3} \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix} \qquad \dot{\mathbf{g}}(x_1, x_2, x_3) = \frac{1}{(x_1 + x_2 + x_3)^2} \begin{bmatrix} x_2 + x_3 & -x_1 \\ x_3 & x_3 \end{bmatrix}$$

$$\therefore \mathbf{g}\left(\frac{S_{k_{1n}}}{n+1}, \frac{S_{k_{2n}} - S_{k_{1n}}}{n+1}, \frac{S_{n+1} - S_{k_{2n}}}{n+1} \right) = \frac{1}{S_{n+1}} \begin{bmatrix} S_{k_{1n}} \\ S_{k_{2n}} \end{bmatrix}$$

which has the same distribution as $\left(U_{(k_{1n})}, U_{(k_{2n})}\right)^{\mathsf{T}}$, by Result 1. By the Delta Method

$$\sqrt{n}\left(\begin{pmatrix} U_{(k_{1n})} \\ U_{(k_{2n})} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}\left(0, \dot{\mathbf{g}}(\mu) \Sigma \dot{\mathbf{g}}(\mu)^{\mathsf{T}}\right)$$

where Σ is as defined in the Result 4, where here $\mu = (p_1, p_2 - p_1, 1 - p_2)^T$. It can be easily verified that

$$\dot{\mathbf{g}}(\mu) \Sigma \dot{\mathbf{g}}(\mu)^T = \begin{bmatrix} p_1(1 - p_1) & p_1(1 - p_2) \\ p_1(1 - p_2) & p_2(1 - p_2) \end{bmatrix}.$$

The Joint Distribution Of Sample Quantiles (cont.)

RESULT 6: If $X_{(1)}, \ldots, X_{(n)}$ are the order statistics from a random sample of size n from a distribution with continuous distribution function F_X and density f_X which is continuous and non-zero in a neighbourhood of quantiles x_{p_1} and x_{p_2} corresponding to probabilities $p_1 < p_2$, then if $k_{1n} = \lceil np_1 \rceil$ and $k_{2n} = \lceil np_2 \rceil$

$$\sqrt{n}\left(\begin{pmatrix} X_{(k_{1n})} \\ X_{(k_{2n})} \end{pmatrix} - \begin{pmatrix} x_{p_1} \\ x_{p_2} \end{pmatrix}\right) \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} \frac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2} & \frac{p_1(1-p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} \\ \frac{p_1(1-p_2)}{f_X(x_{p_1}) f_X(x_{p_2})} & \frac{p_2(1-p_2)}{\{f_X(x_{p_2})\}^2} \end{bmatrix}\right)$$

The Joint Distribution Of Sample Quantiles (cont.)

Proof: We use the Delta Method on the result from Result 5, with the transformation

$$\mathbf{g}(y_1, y_2) = \begin{bmatrix} F_X^{-1}(y_1) \\ F_X^{-1}(y_2) \end{bmatrix}$$

so that

$$\dot{\mathbf{g}}(y_1, y_2) = \begin{bmatrix} \dfrac{1}{f_X\left(F_X^{-1}(y_1)\right)} & 0 \\ 0 & \dfrac{1}{f_X\left(F_X^{-1}(y_2)\right)} \end{bmatrix}$$

with $y_1 = p_1$ and $y_2 = p_2$.

The Joint Distribution Of Sample Quantiles (cont.)

By properties of the multivariate normal distribution, we have that the marginal distribution of $X_{(k_{1n})}$ can be approximated for large n by using the relationship

$$\sqrt{n}(X_{(k_{1n})} - x_{p_1}) \xrightarrow{d} \mathcal{N}\left(0, \frac{p_1(1-p_1)}{\{f_X(x_{p_1})\}^2}\right)$$

For example, if $p_1 = 1/2$, x_{p_1} is the **median** $x_{F_X}(0.5)$ of the distribution, and $X_{(k_{1n})}$ is the **sample median** $\widetilde{X}_n(0.5)$, and we have that

$$\sqrt{n}(\widetilde{X}_n(0.5) - x_{F_X}(0.5)) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4\{f_X(x(0.5))\}^2}\right)$$

If F_X is the $\mathcal{N}(\mu, \sigma^2)$ distribution, then $x_{F_X}(0.5) = \mu$ and

$$f_X(x(0.5)) = f_X(\mu) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/2}$$

The Joint Distribution Of Sample Quantiles (cont.)

so this result says that

$$\sqrt{n}(\widetilde{X}_n(0.5) - \mu) \xrightarrow{d} \mathcal{N}\left(0, \frac{\pi\sigma^2}{2}\right) \approx \mathcal{N}\left(0, 1.57\sigma^2\right)$$

which contrasts with the exact result for the sample mean

$$\sqrt{n}(\overline{X}_n - \mu) \sim \mathcal{N}\left(0, \sigma^2\right).$$