Characteristic Functions and Cumulant Expansions

Expansion of cf and mgf

If the mgf $M_X(t)$ exists for $|t| < \delta$, then

$$\varphi_X(t) = M_X(it)$$

for t in this neighbourhood of zero.

For the cumulants, we have by the previous result.

$$\kappa_{Xj} = \frac{\mathrm{d}^j}{\mathrm{d}t^j} \{\log M_X(t)\}_{t=0}$$

Expansion of cf and mgf (cont.)

Theorem

If the mgf $M_X(t)$ exists for $|t| < \delta$, then for any $j = 1, 2, \ldots$

$$\frac{d^j}{dt^j} \{\log M_X(t)\}_{t=0} = \frac{1}{(i)^j} \frac{d^j}{dt^j} \{\log \varphi_X(t)\}_{t=0}$$

with both equal to $\mathbb{E}_X[X^j]$.

Expansion of cf and mgf (cont.)

Proof.

For each m we have the expansions

$$M_X(t) = 1 + \sum_{j=1}^{m} \frac{t^j}{j!}\mathbb{E}[X^j] + \mathrm{o}(t^m)$$

$$\varphi_X(t) = 1 + \sum_{j=1}^{m} \frac{(it)^j}{j!}\mathbb{E}[X^j] + \mathrm{o}(t^m)$$

as $t \longrightarrow \infty$. But $\varphi_X(t) = M_X(it)$ for each t, $|t| < \delta$, so we must have matching coefficients for the t^j terms.

Lemma

Lemma (Severini p. 113)

If $\alpha(t) > 0$ *has the exponential series expansion around* $t = 0$

$$\alpha(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} t^j$$

and we wish to represent

$$\beta(t) = \log \alpha(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} t^j$$

Then $\alpha_0 = \exp(\beta_0)$ *and*

$$\alpha_{r+1} = \sum_{j=0}^{r} \binom{r}{j} \beta_{j+1} \alpha_{r-j} \qquad r = 0, 1, 2, \ldots$$

Lemma

Proof.

Clearly

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}\{\alpha(t)\}_{t=0} = \alpha_j \qquad \frac{\mathrm{d}^j}{\mathrm{d}t^j}\{\beta(t)\}_{t=0} = \beta_j$$

Also, as $\alpha(t) = \exp\{\beta(t)\}$

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = \exp\{\beta(t)\}\frac{\mathrm{d}\beta(t)}{\mathrm{d}t} = \alpha(t)\frac{\mathrm{d}\beta(t)}{\mathrm{d}t}$$

so therefore using standard calculus for products

$$\frac{\mathrm{d}^{r+1}\alpha(t)}{\mathrm{d}t^{r+1}} = \sum_{j=0}^{r}\binom{r}{j}\frac{\mathrm{d}^{j+1}\beta(t)}{\mathrm{d}t^{j+1}}\frac{\mathrm{d}^{r-j}\alpha(t)}{\mathrm{d}t^{r-j}}$$

and the result follows setting $t = 0$. □

Moments and Cumulants

Therefore consider mgf and cgf expansions

$$M_X(t) = 1 + \sum_{j=1}^{\infty} \frac{t^j}{j!} \mathbb{E}[X^j]$$

$$K_X(t) = \log M_X(t) = 1 + \sum_{j=1}^{\infty} \frac{t^j}{j!} \kappa_{Xj}$$

and regard $M_X(t)$ as $\alpha(t)$ and $K_X(t)$ as $\beta(t)$, we have for $r = 0, 1, 2 \ldots$

$$\mathbb{E}_X[X^{r+1}] = \sum_{j=0}^{r} \binom{r}{j} \kappa_{X,j+1} \mathbb{E}_X[X^{r-j}]$$

yielding a recursion for the moments in terms of the cumulants.

Moments and Cumulants

We therefore have a deterministic relationship between the moments and cumulants.

Expansion of cf

Theorem

If X is real valued with cf $\varphi_X(t)$, and $\mathbb{E}[|X|^m] < \infty$, then

$$\log \varphi_X(t) = \sum_{j=1}^{m} \frac{(it)^j}{j!} \kappa_{Xj} + o(t^m)$$

as $t \longrightarrow \infty$.

Expansion of cf (cont.)

As $\mathbb{E}[|X|^m] < \infty$, $\mathbb{E}[X^m] < \infty$, so we have

$$\varphi_X(t) \quad = \quad 1 + \sum_{j=1}^{m} \frac{(it)^j}{j!} \mathbb{E}[X^j] + \mathrm{o}(t^m)$$

as $t \longrightarrow \infty$. Now for any complex z

$$-\log(1+z) = (-z) + \frac{(-z)^2}{2} + \cdots = \sum_{j=1}^{d} \frac{(-z)^j}{j} + \mathrm{o}(|z|^d)$$

Expansion of cf (cont.)

Therefore

$$\log \varphi_X(t) = \log \left[1 + \sum_{j=1}^{m} \frac{(it)^j}{j!} \mathbb{E}[X^j] + \mathrm{o}(t^m) \right]$$

That is

$$\log \varphi_X(t) \quad = \quad \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j} \left[\sum_{k=1}^{m} \frac{(it)^k}{k!} \mathbb{E}[X^k] + \mathrm{o}(t^m) \right]^j + \mathrm{o}(t^m)$$

which may be expanded to compute individual coefficients.

Expansion of cf (cont.)

More importantly, it follows that the general form of the resulting series is

$$\log \varphi_X(t) = \sum_{j=1}^{m} \frac{(it)^j}{j!} c_j + o(t^m)$$

as $t \longrightarrow 0$ for some coefficients c_j - see footnote.[1]

However, we know from this representation that

$$\frac{d^j}{dt^j} \{\log \varphi_X(t)\}_{t=0} = (i)^j c_j$$

and yet from the earlier result

$$\frac{d^j}{dt^j} \{\log M_X(t)\}_{t=0} = \frac{1}{(i)^j} \frac{d^j}{dt^j} \{\log \varphi_X(t)\}_{t=0}$$

[1] in my notes on the board, I had a j in the denominator not $j!$ - that is still valid, but it doesn't then match the cumulant expansion

Expansion of cf (cont.)

Therefore, we must have that for each j

$$c_j = \kappa_{Xj} \qquad j = 1, 2, \ldots, m.$$