Characteristic Function for discrete random variable

CF calculation

If

$$\varphi_X(t) = \cos(\theta t) \qquad t \in \mathbb{R}$$

what is f_X ?

As

$$\limsup_{|t| \longrightarrow \infty} |\varphi_X(t)| = 1$$

we have that f_X is discrete, so we use the inversion formula

$$f_X(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ixt} \varphi_X(t) \, \mathrm{d}t$$

CF calculation (cont.)

For x in some countable set $\{x_1, x_2, \ldots\}$

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\cos(\theta t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ixt}\frac{(e^{i\theta t}+e^{-i\theta t})}{2}\,\mathrm{d}t \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{-i(x-\theta)t}\,\mathrm{d}t + \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{-i(x+\theta)t}\,\mathrm{d}t \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\cos(-(x-\theta)t) + i\sin(-(x-\theta)t)\}\,\mathrm{d}t \\
&\qquad + \frac{1}{4\pi}\int_{-\pi}^{\pi} \{\cos(-(x+\theta)t) + i\sin(-(x+\theta)t)\}\,\mathrm{d}t
\end{aligned}
$$

CF calculation (cont.)

Integrating term by term for the real part

$$\int_{-\pi}^{\pi} \cos(-(x-\theta)t)\,\mathrm{d}t = \begin{cases} 2\pi & x = \theta \\ \left[-\frac{1}{(x-\theta)}\sin(-(x-\theta)t)\right]_{-\pi}^{\pi} & x \neq \theta \end{cases}$$

$$= \begin{cases} 2\pi & x = \theta \\ \frac{1}{(x-\theta)}\left[\sin((x-\theta)\pi) - \sin(-(x-\theta)\pi)\right] & x \neq \theta \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(-(x+\theta)t)\,\mathrm{d}t = \begin{cases} 2\pi & x = -\theta \\ \left[-\frac{1}{(x+\theta)}\sin(-(x+\theta)t)\right]_{-\pi}^{\pi} & x \neq -\theta \end{cases}$$

$$= \begin{cases} 2\pi & x = - \\ \frac{1}{(x+\theta)}\left[\sin((x+\theta)\pi) - \sin(-(x+\theta)\pi)\right] & x \neq - \end{cases}$$

CF calculation (cont.)

Integrating term by term for the imaginary part

$$\int_{-\pi}^{\pi} \sin(-(x-\theta)t)\,\mathrm{d}t = \begin{cases} 0 & x=\theta \\ \left[\frac{1}{(x-\theta)}\cos(-(x-\theta)t)\right]_{-\pi}^{\pi} & x\neq\theta \end{cases}$$

$$= \begin{cases} 0 & x=\theta \\ \frac{1}{(x-\theta)}\left[\cos(-(x-\theta)\pi)-\cos((x-\theta)\pi)\right] & x\neq\theta \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(-(x+\theta)t)\,\mathrm{d}t = \begin{cases} 0 & x=-\theta \\ \left[\frac{1}{(x+\theta)}\cos((x+\theta)t)\right]_{-\pi}^{\pi} & x\neq-\theta \end{cases}$$

$$= \begin{cases} 0 & x=- \\ \frac{1}{(x+\theta)}\left[\cos(-(x+\theta)\pi)-\cos((x+\theta)\pi)\right] & x\neq- \end{cases}$$

CF calculation (cont.)

Thus for $x \neq \pm\theta$, the imaginary part is

$$\frac{1}{(x-\theta)}\left[\cos(-(x-\theta)\pi) - \cos((x-\theta)\pi)\right] + \frac{1}{(x+\theta)}\left[\cos(-(x+\theta)\pi) - \cos((x+\theta)\pi)\right]$$

This is non-zero for all such x. Thus, to have a purely real result, we can consider $x = \pm\theta$ only, in which case

$$f_X(x) = \frac{1}{2} \qquad x = \pm\theta$$

and zero otherwise.