$$|\varphi_X(t)|^2 = 1 \ \forall t \quad \Longleftrightarrow \quad X = c \text{ w.p. } 1$$

Set-up from A2-Q2

Given the independence statements, we are able to deduce that

$$\varphi_{X_2}(t) = \varphi_{X_2}(t)\varphi_{X_1}(t)\varphi_{X_1}(-t) = \varphi_{X_2}(t)|\varphi_{X_1}(t)|^2$$

But $\varphi_{X_2}(0) = 1$, and thus by continuity

$$\varphi_{X_2}(t) \neq 0 \qquad |t| < \delta$$

for some δ small enough. Thus, for at least $|t| < \delta$,

$$|\varphi_{X_1}(t)|^2 = 1.$$

Set-up from A2-Q2 (cont.)

Let $t_0 \in (-\delta, \delta)$. Then we have that

$$\varphi_{X_1}(t_0) = e^{it_0 c} = \cos(ct) + i\sin(ct)$$

for some c, as this is the general form of a complex number on the unit circle. That is

$$\varphi_{X_1}(t_0) = \int e^{it_0 x} \mathrm{d}F_X(x) = e^{it_0 c}$$

or equivalently

$$\int (e^{it_0 x} - e^{it_0 c}) \mathrm{d}F_X(x) = 0.$$

Set-up from A2-Q2 (cont.)

We may rewrite this

$$\int e^{it_0 c}(1 - e^{it_0(x-c)})\mathrm{d}F_{X_1}(x) = 0.$$

But $e^{it_0 c}$ is a constant, non-zero term, so this equation simplifies to

$$\int (1 - e^{it_0(x-c)})\mathrm{d}F_{X_1}(x) = 0.$$

Taking the real part of this expression

$$\int (1 - \cos(t_0(x-c)))\mathrm{d}F_{X_1}(x) = 0.$$

Set-up from A2-Q2 (cont.)

Now the function $(1 - \cos(t_0(x - c)))$ is non-negative for all x; thus for the integral to be zero, we must have that the set of zeros of the function

$$1 - \cos(t_0(x - c))$$

includes as a subset the collection of x values where $\mathrm{d}F_{X_1}(x)$ is non-zero.

The function $1 - \cos(t_0(x - c))$ has a countable number of zeros at

$$c + \frac{2j\pi}{t_0} \qquad j \in \mathbb{Z}$$

Therefore the support of the distribution F_{X_1} is countable, and thus X_1 is discrete.

Set-up from A2-Q2 (cont.)

Now consider $t_1 \in (-\delta, \delta)$, $t_1 \neq kt_0$ for any integer k. Following through the same logic, we must also have that the support is included in the collection of x values

$$c + \frac{2j\pi}{t_1} \qquad j \in \mathbb{Z}$$

Thus the entire support must be the single point c, as this point is the entire intersection of the two sets

$$\left\{ c + \frac{2j\pi}{t_0} \qquad j \in \mathbb{Z} \right\} \qquad \left\{ c + \frac{2j\pi}{t_1} \qquad j \in \mathbb{Z} \right\}$$

Thus X_1 is degenerate at c.