MATH 556 - EXERCISES 3: SOLUTIONS

1 (a) This is not an Exponential Family distribution; the support is parameter dependent.

(b) This is an EF distribution with $m = 1$:

$$f(x;\theta) = \frac{\mathbb{I}_{\{1,2,3,\ldots\}}(x)}{x} \frac{-1}{\log(1-\theta)} \exp\{x \log\theta\} = \exp\{c(\theta)T(x) - A(\theta)\}h(x)$$

where $\quad h(x) = \dfrac{\mathbb{I}_{\{1,2,3,\ldots\}}(x)}{x} \qquad A(\theta) = \log(-\log(1-\theta)) \qquad c(\theta) = \log(\theta) \qquad T(x) = x,$

so the natural parameter is $\eta = \log(\theta)$.

(c) This is an EF distribution with $k = 2$:

$$\begin{aligned} f(x;\phi,\lambda) &= \frac{\mathbb{I}_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \sqrt{\lambda} e^{\phi} \exp\left\{-\frac{\phi^2}{2\lambda}x - \frac{\lambda}{2}\frac{1}{x}\right\} \\ &= \exp\{c_1(\phi,\lambda)T_1(x) + c_2(\phi,\lambda)T_2(x) - A(\phi,\lambda)\}h(x) \end{aligned}$$

where

$$h(x) = \frac{\mathbb{I}_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \qquad A(\phi,\lambda) = -\phi - \frac{1}{2}\log\lambda$$

and

$$c_1(\phi,\lambda) = -\frac{\phi^2}{2\lambda} \qquad c_2(\phi,\lambda) = -\frac{\lambda}{2} \qquad T_1(x) = x \qquad T_2(x) = \frac{1}{x},$$

so the natural parameter is $\eta = (\eta_1, \eta_2)^\top$ where

$$\eta_1 = -\phi^2/2\lambda \qquad \eta_2 = -\lambda/2.$$

Thus

$$\phi = 2\sqrt{\eta_1\eta_2} \qquad \lambda = -2\eta_2.$$

In the natural parameterization, therefore,

$$k(\eta_1,\eta_2) = -2\sqrt{\eta_1\eta_2} - \frac{1}{2}\log(-2\eta_2)$$

and using the results from lectures

$$\mathbb{E}_{f_X}[1/X] = \mathbb{E}_{f_X}[T_2(X)] = \frac{\partial k(\eta_1,\eta_2)}{\partial\eta_2}.$$

We have

$$\mathbb{E}_{f_X}[1/X] = \frac{\partial}{\partial\eta_2}\left\{-2\sqrt{\eta_1\eta_2} - \frac{1}{2}\log(-2\eta_2)\right\} = \left\{-\sqrt{\eta_1/\eta_2} - \frac{1}{2\eta_2}\right\} = \frac{\phi}{\lambda} + \frac{1}{\lambda}$$

2 (a) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$ and $0 \le t \le 1$. Then

$$\begin{aligned} \int h(x) e^{(t\eta_1 + (1-t)\eta_2)^\top T(x)}\, \mathrm{d}x &= \int h(x) e^{(t\eta_1)^\top T(x)} e^{((1-t)\eta_2)^\top T(x)}\, \mathrm{d}x \\ &\le \left\{\int h(x) e^{(t\eta_1)^\top T(x)}\, \mathrm{d}x\right\}\left\{\int h(x) e^{((1-t)\eta_2)^\top T(x)}\, \mathrm{d}x\right\} \\ &\le \left\{\int h(x) e^{\eta_1^\top T(x)}\, \mathrm{d}x\right\}^t \left\{\int h(x) e^{\eta_2^\top T(x)}\, \mathrm{d}x\right\}^{(1-t)} < \infty \end{aligned}$$

so $t\eta_1 + (1-t)\eta_2 \in \mathcal{H}$.

(b) We can re-write f_X as

$$f_X(x;\eta) = \exp\left\{\eta^\top T(x) - k(\eta)\right\} h(x)$$

and by integrating with respect to x, we note that

$$\int h(x)\exp\left\{\eta T(x)\right\} \; \mathrm{d}x = \exp\{k(\eta)\}$$

for $\eta \in \mathcal{H}$ as given in lectures. Thus, for s in a suitable neighbourhood of zero, $||s|| < \delta$ say, we have

$$\begin{aligned} M_T(s) &= \mathbb{E}_{f_X}[e^{s^\top T(X)}] = \int e^{s^\top T(x)} h(x)\exp\left\{\eta^\top T(x) - k(\eta)\right\} \; \mathrm{d}x \\ &= \exp\{-k(\eta)\}\int h(x)\exp\left\{(\eta+s)^\top T(x)\right\} \; \mathrm{d}x = \exp\{-k(\eta)\}\exp\{k(\eta+s)\} \end{aligned}$$

as $\eta \in \mathcal{H} \Longrightarrow \eta + s \in \mathcal{H}$ for s small enough, as $\mathcal{H}$ is open. Hence, as $K_T(s) = \log M_T(s)$,

$$K_T(s) = k(\eta + s) - k(\eta)$$

for $||s|| < \delta$ as required.

(c) By inspection

$$\ell(x;\eta_1,\eta_2) = (\eta_1 - \eta_2)^\top T(x) - (k(\eta_1) - k(\eta_2))$$