MATH 556 - EXERCISES 2: SOLUTIONS

1. Using the Chebychev Lemma with $h(x) = e^{tx}$ and $c = e^{at}$, for $t > 0$,

$$\mathrm{P}_X\left[X \geq a\right] = \mathrm{P}_X\left[tX \geq at\right] = \mathrm{P}_X\left[\exp\{tX\} \geq \exp\{at\}\right] \leq \frac{\mathbb{E}_X[e^{tX}]}{e^{at}} = \frac{M_X(t)}{e^{at}}$$

provided $t < \delta$ also. Using similar methods,

$$\mathrm{P}_X\left[X \leq a\right] \leq e^{-at} M_X(t) \qquad \text{for } -\delta < t < 0$$

2. We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds}\left\{K_X(t)\right\}_{s=t} = \frac{d}{ds}\left\{\log M_X(t)\right\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \Longrightarrow K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = \mathbb{E}_X[\, X\,]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \left\{M_X^{(1)}(t)\right\}^2}{\left\{M_X(t)\right\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \left\{M_X^{(1)}(0)\right\}^2}{\left\{M_X(0)\right\}^2} = \mathbb{E}_X[\, X^2\,] - \{\mathbb{E}_X[\, X\,]\}^2$$

3. (a) From first principles, for $y > 0$,

$$\mathrm{P}[Y \leq y] = \mathrm{P}_X[X^2 \leq y] = \mathrm{P}_X[-\sqrt{y} \leq X \leq \sqrt{y}] = \Phi(\sqrt{y} - \mu) - \Phi(-\sqrt{y} - \mu)$$

therefore for $y > 0$,

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}}\left(\phi(\sqrt{y} - \mu) + \phi(-\sqrt{y} - \mu)\right) \\ &= \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\left(\exp\left\{-\frac{1}{2}(\sqrt{y} - \mu)^2\right\} + \exp\left\{-\frac{1}{2}(-\sqrt{y} - \mu)^2\right\}\right) \\ &= \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{1}{2}(y + \mu^2)\right\}\left(\exp\left\{\sqrt{y}\mu\right\} + \exp\left\{-\sqrt{y}\mu\right\}\right) \end{aligned}$$

Let $\delta = \mu^2$. We rewrite the density

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{1}{2}(y + \delta)\right\}\left(\exp\left\{\sqrt{y\delta}\right\} + \exp\left\{-\sqrt{y\delta}\right\}\right)$$

Using the exponential series expansion, we have that

$$\begin{aligned} \exp\left\{\sqrt{y\delta}\right\} + \exp\left\{-\sqrt{y\delta}\right\} &= \sum_{j=0}^{\infty}\frac{1}{j!}y^{j/2}\delta^{j/2} - \sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j y^{j/2}\delta^{j/2} \\ &= 2\sum_{j=0}^{\infty}\frac{(\delta y)^j}{(2j)!} \end{aligned}$$

Therefore the density is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{1}{2}(y+\delta)\right\}\sum_{j=0}^{\infty}\frac{(\delta y)^j}{(2j)!}$$

Rewriting this, we have for $y > 0$,

$$\begin{aligned} f_Y(y) &= \exp\left\{-\frac{\delta}{2}\right\}\sum_{j=0}^{\infty}\frac{(\delta y)^j}{(2j)!}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}\exp\left\{-\frac{y}{2}\right\} \\ &= \exp\left\{-\frac{\delta}{2}\right\}\sum_{j=0}^{\infty}\frac{\delta^j}{(2j)!}\frac{1}{\sqrt{2\pi}}y^{j-1/2}\exp\left\{-\frac{y}{2}\right\} \\ &= \exp\left\{-\frac{\delta}{2}\right\}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{(2j)!}\frac{2^{j+1/2}}{\sqrt{\pi}}y^{j-1/2}\exp\left\{-\frac{y}{2}\right\} \\ &= \exp\left\{-\frac{\delta}{2}\right\}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{\Gamma(2j+1)}\frac{2^{j+1/2}}{\sqrt{\pi}}y^{j-1/2}\exp\left\{-\frac{y}{2}\right\} \end{aligned}$$

Note that

$$\Gamma(2j+1) = (2j)! = (2j)(2j-1)(2j-2)....3.2.1 = 2^{2j}j(j-1/2)(j-1)...(3/2)(2/2)(1/2).$$

Now, $Z_m \sim \chi^2_m \equiv \text{Gamma}(m/2, 1/2)$, we have

$$f_{Z_m}(z) = \frac{(1/2)^{m/2}}{\Gamma(m/2)}z^{m/2-1}\exp\{-z/2\} \qquad z > 0$$

so if $m = 2j + k$,

$$f_{Z_{2j+k}}(z) = \frac{(1/2)^{j+k/2}}{\Gamma(j+k/2)}z^{j+k/2-1}\exp\{-z/2\} \qquad z > 0$$

Therefore we have after some cancellation,

$$f_Y(y) = e^{-\delta/2}\sum_{j=0}^{\infty}\frac{(\delta/2)^j}{j!}f_{Z_{2j+k}}(y) \qquad y > 0$$

with $k = 1$.

(b) We have for $\varphi_Y(t)$, from the definition

$$\varphi_Y(t) = \mathbb{E}_Y[e^{itY}] = \mathbb{E}_X[e^{itX^2}] = \int_{-\infty}^{\infty}e^{itx^2}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(x-\mu)^2\right\}\,\mathrm{d}x$$

We may complete the square in the integral to obtain

$$\varphi_Y(t) = \exp\left\{\frac{\mu^2 it}{1-2it}\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(1-2it)}{2}\left(x-\frac{\mu}{(1-2it)}\right)^2\right\}\,\mathrm{d}x$$

and thus, integrating the normal kernel,

$$\varphi_Y(t) = \frac{1}{\sqrt{1-2it}}\exp\left\{\frac{\mu^2 it}{1-2it}\right\}$$

(here as the integral is wrt x, we may just treat the quantity i as if it were a real quantity during the manipulation).

(c) We have

$$\mathcal{L}_Y(t) = \varphi_Y(it) = \frac{1}{\sqrt{1+2t}} \exp\left\{-\frac{\mu^2 t}{1+2t}\right\}$$

(d) In this case the mgf exists, and

$$M_Y(t) = \varphi_Y(-it) = \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\mu^2 t}{1-2t}\right\}$$

so

$$K_Y(t) = -\frac{1}{2}\log(1-2t) + \frac{\mu^2 t}{1-2t}.$$

Thus

$$\begin{aligned}
\mathbb{E}_Y[Y] = \frac{d}{dt}\{K_Y(t)\}_{t=0} &= \left\{\frac{1}{1-2t} + \frac{(1-2t)\mu^2 + 2\mu^2 t}{(1-2t)^2}\right\}_{t=0} \\
&= \left\{\frac{1}{1-2t} + \frac{\mu^2}{(1-2t)^2}\right\}_{t=0} \\
&= 1+\mu^2
\end{aligned}$$

and

$$\mathrm{Var}_Y[Y] = \frac{d^2}{dt^2}\{K_Y(t)\}_{t=0} = \left\{\frac{2}{(1-2t)^2} + \frac{4\mu^2}{(1-2t)^3}\right\}_{t=0} = 2+4\mu^2$$

(e) Using mgfs, we have for integers ν_i, $i = 1, \ldots, n$,

$$M_S(t) = \left(\frac{1}{1-2t}\right)^{\nu/2} \exp\left\{\frac{\lambda t}{1-2t}\right\}$$

where

$$\nu = \sum_{i=1}^n \nu_i \qquad \lambda = \sum_{i=1}^n \lambda_i$$

To see this, note that if $n = 2$ with $\nu_1 = \nu_2 = 1$, but $\lambda_1 = \mu_1^2$ and $\lambda_2 = \mu_2^2$, we have

$$\begin{aligned}
M_S(t) = M_{Y_1}(t)M_{Y_2}(t) &= \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\lambda_1 t}{1-2t}\right\} \frac{1}{\sqrt{1-2t}} \exp\left\{\frac{\lambda_2 t}{1-2t}\right\} \\
&= \left(\frac{1}{\sqrt{1-2t}}\right)^2 \exp\left\{\frac{(\lambda_1+\lambda_2)t}{1-2t}\right\}
\end{aligned}$$

We can use this as a recursion to generate the mgf for any integer ν_1.

Thus S has a noncentral chi-square distribution with ν degrees of freedom, and noncentrality λ.

4. We have

$$\mathcal{L}_X(t) = \int e^{-tx} dF_X(x)$$

and so differentiating under the integral wrt t, we have

$$\frac{d^r}{dt^r}\mathcal{L}_X(t) = \int \left\{\frac{d^r}{dt^r} e^{-tx}\right\} \, \mathrm{d}F_X(x) = \int (-x)^r e^{-tx} \, \mathrm{d}F_X(x) = (-1)^r \int x^r e^{-tx} \, \mathrm{d}F_X(x)$$

and the result follows multiplying both sides by $(-1)^r$, and recalling that the variable is nonnegative.

5. We have, integrating by parts

$$\begin{aligned}\mathcal{L}_X(t) &= \int_0^\infty e^{-tx} f_X(x)\,\mathrm{d}x \\ &= \left[e^{-tx}F_X(x)\right]_0^\infty + t\int_0^\infty e^{-tx}F_X(x)\,\mathrm{d}x \\ &= t\int_0^\infty e^{-tx}F_X(x)\,\mathrm{d}x.\end{aligned}$$

6. The mgf of the Gamma is

$$\left(\frac{\beta}{\beta - t}\right)^\alpha$$

and hence the mgf of Y must be

$$\left(\frac{\beta_1}{\beta_1 - t}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + t}\right)^{\alpha_2}$$

thus the characteristic function

$$\left(\frac{\beta_1}{\beta_1 - it}\right)^{\alpha_1}\left(\frac{\beta_2}{\beta_2 + it}\right)^{\alpha_2}$$

In principle we could invert this cf using the inversion formula for continuous rvs. However, the density is not straightforward.

The difference of two Gamma random variables has the *Variance Gamma* distribution.