MATH 556 - EXERCISES 3
These exercises are not for assessment

1. State whether each of the following functions defines an Exponential Family distribution. Where it is possible, write the distribution in the Exponential Family form, and find the natural (canonical) parameterization. If the function does not specify an Exponential Family distribution, explain why not.

   (a) The continuous $Uniform(\theta_1, \theta_2)$ distribution:

   $$f_X(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \qquad \theta_1 < x < \theta_2$$

   and zero otherwise, for parameters $\theta_1 < \theta_2$.

   (b) The distribution defined by

   $$f_X(x; \theta) = \frac{-1}{\log(1-\theta)} \frac{\theta^x}{x} \qquad x = 1, 2, 3, \ldots$$

   and zero otherwise, for parameter θ, where $0 < \theta < 1$.

   (c) The distribution defined by

   $$f_X(x; \phi, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\phi^2}{2\lambda}x + \phi - \frac{\lambda}{2x}\right\} \qquad x > 0$$

   and zero otherwise, for parameters $\phi, \lambda > 0$. Find the expectation of $1/X$ in terms of ϕ and λ.

2. For scalar random variable X, consider a one parameter Exponential Family distribution in its natural parameterization,

   $$f_X(x; \eta) = \exp\{\eta\, T(x) - k(\eta)\}\, h(x)$$

   and natural parameter space $\mathcal{H}$. Suppose that $\mathcal{H}$ is an open interval in $\mathbb{R}$, so that for every $\eta \in \mathcal{H}$, there exists an $\epsilon > 0$ such that

   $$\eta' \in \mathcal{H} \quad \text{if} \quad |\eta - \eta'| < \epsilon$$

   (a) Show that the natural parameter space $\mathcal{H}$ is a convex set, that is

   $$\eta_1, \eta_2 \in \mathcal{H} \qquad \Longrightarrow \qquad \lambda\eta_1 + (1-\lambda)\eta_2 \in \mathcal{H}$$

   for $0 \leq \lambda \leq 1$.

   (b) Prove that the cumulant generating function of random variable $T = T(X)$ under the probability model f_X takes the form

   $$K_T(t) = \kappa(\eta + t) - \kappa(\eta)$$

   for $t \in (-\delta, \delta)$, some $\delta > 0$, where κ is some function to be identified.

   (c) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$. Find the form of the log likelihood ratio, $\ell(x; \eta_1, \eta_2)$, where

   $$\ell(x; \eta_1, \eta_2) = \log \frac{f_X(x; \eta_1)}{f_X(x; \eta_2)}.$$