MATH 556 - ASSIGNMENT 3

To be handed in not later than 10pm, 14th November 2014.
Please submit your solutions as pdf via myCourses

1. The entropy of a discrete distribution with countable support $\mathfrak{X} = \{x_1, x_2, \ldots\}$ and mass function $f(x)$ is defined by
$$H(f) = -\sum_{x \in \mathfrak{X}} f(x) \log f(x)$$
For this question, suppose $\mathfrak{X} \equiv \{0, 1, 2, \ldots\}$.

   (a) Find the mass function f_X that has the largest possible entropy on this support, subject to the constraint that
$$\mathbb{E}_X[X] = \sum_{x=0}^{\infty} x f_X(x) = \mu$$
6 Marks

   Use Lagrange multipliers to perform the constrained optimization as follows:
   - Denote $f_X(x) = p_x$ for each x;
   - Consider the objective function $\Lambda(.)$ expressed in terms of $\mathbf{p} = (p_0, p_1, \ldots)$ as
$$\Lambda(\mathbf{p}, \lambda_0, \lambda_1) = H(\mathbf{p}) + \lambda_0 \left(\sum_{x=0}^{\infty} p_x - 1 \right) + \lambda_1 \left(\sum_{x=0}^{\infty} x p_x - \mu \right)$$
where the first term is the entropy defined for the distribution specified by $\mathbf{p}$, the second term acknowledges the constraint that the probabilities must sum to one, and the third term acknowledges the constraint that the expectation must equal μ;
   - maximize $\Lambda(\mathbf{p}, \lambda_0, \lambda_1)$ using calculus methods.

   (b) Suppose now that m constraints of the form
$$\mathbb{E}_X[g_k(X)] = \omega_k \qquad k = 1, \ldots, m$$
are placed on the distribution. Show that the distribution that maximizes entropy on this support is an Exponential Family distribution.

4 Marks

2. Consider the discrete distribution for random variable X defined on $\{0, 1, 2, \ldots\}$ with mass function
$$f_X(x; \theta) = \frac{h(x)\theta^x}{\sum\limits_{y=0}^{\infty} h(y)\theta^y} \qquad x = 0, 1, 2, \ldots$$
with $\theta > 0$. Verify that this is an Exponential Family distribution, and also that if $X_1, \ldots, X_n$ are independent random variables having this distribution, then the distribution of random variable
$$S_n = \sum_{i=1}^{n} X_i$$
is also a member of the same distributional family.

5 Marks

3. Consider the continuous random variable X with density function

$$f_X(x;\mu,\lambda) = C(\mu,\lambda)\exp\{-\lambda(x-\mu)^3\} \quad x \in \mathbb{R}$$

for parameters $\mu \in \mathbb{R}, \lambda > 0$. Is this a location-scale family distribution, an Exponential Family distribution, or both ? Justify your answer.

5 Marks