MATH 556 - ASSIGNMENT 3: SOLUTIONS

1. (a) We have

$$\Lambda(\mathbf{p}, \lambda_0, \lambda_1) = -\sum_{x=0}^{\infty} p_x \log p_x + \lambda_0 \left(\sum_{x=0}^{\infty} p_x - 1 \right) + \lambda_1 \left(\sum_{x=0}^{\infty} x p_x - \mu \right)$$

- by convention, and without loss of generality, we assume $\lambda_0, \lambda_1 > 0$. Differentiating wrt p_x and equating to zero yields

$$-(1 + \log p_x) + \lambda_0 + \lambda_1 x = 0$$

that is, for $x = 0, 1, 2, \ldots$.

$$p_x = \exp\{-(\lambda_0 - 1) - \lambda_1 x\} \propto \exp\{-\lambda_1 x\}.$$

By the sum to 1 constraint, we must have that

$$p_x = \frac{\exp\{-\lambda_1 x\}}{\sum\limits_{y=0}^{\infty} \exp\{-\lambda_1 y\}} = \frac{\theta^x}{\sum\limits_{y=0}^{\infty} \theta^y}$$

after writing $\theta = \exp\{-\lambda\}$, where $0 < \theta < 1$. Hence, summing the geometric progression in the denominator, we must have

$$p_x = (1 - \theta)\theta^x \qquad x = 0, 1, \ldots.$$

To meet the second constraint, we now see that we must have

$$\sum_{x=1}^{\infty} x(1 - \theta)\theta^x = \mu$$

that is,

$$\mu = \theta(1 - \theta) \sum_{x=1}^{\infty} x\theta^{x-1} = \theta(1 - \theta) \frac{d}{d\theta} \left\{ \sum_{x=1}^{\infty} \theta^x \right\} = \theta(1 - \theta) \frac{d}{d\theta} \left\{ \frac{1}{1 - \theta} - 1 \right\} = \frac{\theta}{1 - \theta}.$$

that is

$$\theta = \frac{\mu}{1 + \mu}$$

6 Marks

(b) We now have

$$\Lambda(\mathbf{p}, \lambda) = -\sum_{x=0}^{\infty} p_x \log p_x + \lambda_0 \left(\sum_{x=0}^{\infty} p_x - 1 \right) + \sum_{k=1}^{m} \lambda_k \left(\sum_{x=0}^{\infty} g_k(x) p_x - \omega_k \right)$$

Using the same logic, differentiation wrt p_x yields for $x = 0, 1, \ldots,$

$$-(1 + \log p_x) + \lambda_0 + \sum_{k=1}^{m} \lambda_k g_k(x) = 0$$

yielding

$$p_x = \exp\left\{ (\lambda_0 - 1) + \sum_{k=1}^{m} \lambda_k g_k(x) \right\} \propto \exp\left\{ \sum_{k=1}^{m} \lambda_k g_k(x) \right\}$$

which defines an Exponential Family distribution with statistic $T(x) = (g_1(x), \ldots, g_m(x))^\top$ and natural parameter $\eta = (\lambda_1, \ldots, \lambda_k)^\top$.

4 Marks

2. It is clear that

$$f_X(x;\theta) = \exp\{x\log\theta - A(\theta)\}h(x) \qquad x = 0,1,2,\ldots$$

where

$$A(\theta) = \log\left(\sum_{y=0}^{\infty} h(y)\theta^y\right).$$

so this is a one parameter Exponential Family distribution with natural parameter $\eta = \log\theta$.

2 Marks

The joint distribution of $X = (X_1,\ldots,X_n)$ takes the form

$$f_X(x;\theta) = \exp\left\{\left(\sum_{i=1}^n x_i\right)\log\theta - nA(\theta)\right\}\left\{\prod_{i=1}^n h(x_i)\right\} \qquad x = 0,1,2,\ldots$$

Thus the joint distribution of (X, S_n) is

$$f_{X,S_n}(x,s;\theta) = \exp\{s\log\theta - nA(\theta)\}\left\{\prod_{i=1}^n h(x_i)\right\}\mathbb{1}_s\left(\sum_{i=1}^n x_i\right)$$

and, marginalizing over the distribution of x we have that

$$f_{S_n}(s;\theta) = \exp\{s\log\theta - nA(\theta)\}\, h_{S_n}(s) \qquad x = 0,1,2,\ldots$$

which is of the same form as the distribution of X_i, for $i = 1,\ldots,n$.

3 Marks

3. (I) **Range $\mathbb{R}^+$.** In this case, the distribution is an Exponential Family distribution with

$$T(x) = (x, x^2, x^3)^\top \qquad \theta = (-3\lambda\mu^2, 3\lambda\mu, \lambda)^\top \qquad A(\theta) = -\log C(\mu,\lambda) + \lambda\mu^3$$

and the support does not depend on θ. However, it is not a location scale distribution, as if we make a location scale transformation

$$Y = a + bX$$

we have for $y > a$

$$f_Y(y;\mu,\lambda,a,b) = C(\mu,\lambda)\frac{1}{b}\exp\left\{-\lambda\left(\frac{y-a}{b} - \mu\right)^3\right\} = C(\mu,\lambda)\frac{1}{b}\exp\left\{-\frac{\lambda}{b^3}(y-a-b\mu)^3\right\}$$

This is no longer a two parameter model, as we need to know a to define the support.

The model, is, however, a scale distribution, as X and $Y = bX$ have the same form with λ updated to λ/b^3 after the transform.

(II) **Range (μ,∞).** In this case, the model is not an Exponential Family distribution unless parameter μ is treated as known, as the support depends on μ. However, it is a location scale family, as if we again take $Y = a + bX$, we have for $y > a + b\mu$

$$f_Y(y;\mu,\lambda) = C(\mu,\lambda)\frac{1}{b}\exp\left\{-\lambda\left(\frac{y-a}{b} - \mu\right)^3\right\} = C(\mu,\lambda)\frac{1}{b}\exp\left\{-\frac{\lambda}{b^3}(y-a-b\mu)^3\right\}$$

that is

$$f_Y(y;\mu^*,\lambda^*) = C(\mu^*,\lambda^*)\exp\{-\lambda^*(x-\mu^*)^3\}$$

with $\mu^* = a + b\mu$, $\lambda^* = \lambda/b^3$.

5 Marks