

## 556: MATHEMATICAL STATISTICS I

### DISTRIBUTIONS DERIVED FROM NORMAL RANDOM SAMPLES

Suppose that  $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$  form a random sample.

(a) By the univariate transformation result

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

(b) By the multivariate transformation result

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}} \sim Student(n-1)$$

as  $T$  can be written  $T = Z/\sqrt{V/\nu}$ , where  $Z \sim \mathcal{N}(0, 1)$  and  $V \sim \chi_\nu^2$  are independent random variables defined by

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \quad V = \frac{(n-1)s^2}{\sigma^2}$$

(c) **Fisher-F distribution:** By the multivariate transformation result, if  $V_1 \sim \chi_{k_1}^2$  and  $V_2 \sim \chi_{k_2}^2$  are independent random variables, then

$$Q = \frac{V_1/k_1}{V_2/k_2}$$

has pdf

$$f_Q(x) = \frac{\Gamma\left(\frac{k_1 + k_2}{2}\right)}{\Gamma\left(\frac{k_1}{2}\right)\Gamma\left(\frac{k_2}{2}\right)} \left(\frac{k_1}{k_2}\right)^{k_1/2} x^{k_1/2-1} \left(1 + \frac{k_1}{k_2}x\right)^{-(k_1+k_2)/2} \quad x > 0$$

and zero otherwise. We say that  $Q \sim Fisher(k_1, k_2)$ .

Now, if  $X_1, \dots, X_{n_X} \sim \mathcal{N}(\mu_X, \sigma_X^2)$  and  $Y_1, \dots, Y_{n_Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$  are independent random samples from different distributions, then as

$$\frac{(n_X - 1)s_X^2}{\sigma_X^2} \sim \chi_{n_X-1}^2 \quad \frac{(n_Y - 1)s_Y^2}{\sigma_Y^2} \sim \chi_{n_Y-1}^2$$

it follows that

$$\frac{s_X^2/\sigma_X^2}{s_Y^2/\sigma_Y^2} \sim Fisher(n_X - 1, n_Y - 1)$$

Note that if  $X \sim Fisher(k_1, k_2)$ , then

$$\mathbb{E}_{f_X}[X] = \frac{k_2}{k_2 - 2} \quad (\text{if } k_2 > 2) \quad \text{Var}_{f_X}[X] = 2 \left(\frac{k_2}{k_2 - 2}\right)^2 \frac{(k_1 + k_2 - 2)}{k_1(k_2 - 4)} \quad (\text{if } k_2 > 4)$$

Also

$$X \sim Fisher(k_1, k_2) \implies \frac{1}{X} \sim Fisher(k_2, k_1)$$

$$X \sim Student(k_1) \implies X^2 \sim Fisher(1, k_1)$$

$$X \sim Fisher(k_1, k_2) \implies \frac{(k_1/k_2)X}{1 + (k_1/k_2)X} \sim Beta\left(\frac{k_1}{2}, \frac{k_2}{2}\right)$$