MATH 556 - EXERCISES 4: SOLUTIONS

1 (a) This is not an Exponential Family distribution; the support is parameter dependent.

(b) This is an EF distribution with $k = 1$:

$$f(x|\theta) = \frac{I_{\{1,2,3,\ldots\}}(x)}{x} \frac{-1}{\log(1-\theta)} \exp\{x \log\theta\} = h(x)c(\theta)\exp\{w(\theta)t(x)\}$$

where $$h(x) = \frac{I_{\{1,2,3,\ldots\}}(x)}{x} \qquad c(\theta) = \frac{-1}{\log(1-\theta)} \qquad w(\theta) = \log(\theta) \qquad t(x) = x,$$

so the natural parameter is $\eta = \log(\theta)$.

(c) This is an EF distribution with $k = 2$:

$$\begin{aligned} f(x|\phi,\lambda) &= \frac{I_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \sqrt{\lambda} e^{\phi} \exp\left\{-\frac{\phi^2}{2\lambda}x - \frac{\lambda}{2}\frac{1}{x}\right\} \\ &= h(x)c(\phi,\lambda)\exp\{w_1(\phi,\lambda)t_1(x) + w_2(\phi,\lambda)t_2(x)\} \end{aligned}$$

where
$$h(x) = \frac{I_{(0,\infty)}(x)}{(2\pi x^3)^{1/2}} \qquad c(\phi,\lambda) = \sqrt{\lambda}e^{\phi}$$

and
$$w_1(\phi,\lambda) = -\frac{\phi^2}{2\lambda} \qquad w_2(\phi,\lambda) = -\frac{\lambda}{2} \qquad t_1(x) = x \qquad t_2(x) = \frac{1}{x},$$

so the natural parameter is $\underset{\sim}{\eta} = (\eta_1, \eta_2)^\top$ where

$$\eta_1 = -\phi^2/2\lambda \qquad \eta_2 = -\lambda/2$$

In the natural parameterization

$$c^\star(\eta_1,\eta_2) = \sqrt{-2\eta_2}\exp\{2\sqrt{\eta_1\eta_2}\}$$

so, using the results from lectures

$$\mathbb{E}_{f_X}[1/X] = \mathbb{E}_{f_X}[t_2(X)] = -\frac{\partial}{\partial\eta_2}\log c^\star(\eta_1,\eta_2).$$

We have
$$\log c^\star(\eta_1,\eta_2) = \frac{1}{2}\log(-2\eta_2) + 2\sqrt{\eta_1\eta_2}$$

and hence

$$\begin{aligned} \mathbb{E}_{f_X}[1/X] &= -\frac{\partial}{\partial\eta_2}\left\{\frac{1}{2}\log(-2\eta_2) + 2\sqrt{\eta_1\eta_2}\right\} = -\left\{\frac{1}{2}\frac{1}{-2\eta_2}(-2) + 2\sqrt{\frac{\eta_1}{\eta_2}}\frac{1}{2}\right\} \\ &= -\frac{1}{2\eta_2} - \sqrt{\frac{\eta_1}{\eta_2}} = \frac{1}{\lambda} + \frac{\phi}{\lambda} \end{aligned}$$

2 (a) Suppose that $\eta_1, \eta_2 \in \mathcal{H}$ and $0 \le \lambda \le 1$. Then

$$\begin{aligned}
\int h(x) e^{(\lambda\eta_1 + (1-\lambda)\eta_2)\, t(x)}\, dx &= \int h(x) e^{\lambda\eta_1 t(x)} e^{(1-\lambda)\eta_2 t(x)}\, dx \\
&\le \left\{\int h(x) e^{\lambda\eta_1 t(x)} dx\right\}\left\{\int h(x) e^{(1-\lambda)\eta_2 t(x)}\, dx\right\} \\
&\le \left\{\int h(x) e^{\eta_1 t(x)} dx\right\}^{\lambda}\left\{\int h(x) e^{\eta_2 t(x)}\, dx\right\}^{(1-\lambda)} \\
&< \infty
\end{aligned}$$

so $\lambda\eta_1 + (1-\lambda)\eta_2 \in \mathcal{H}$.

(b) We can re-write f_X as

$$f_X(x|\eta) = h(x) \exp\{\eta t(x) - \kappa(\eta)\}$$

where $\kappa(\eta) = -\log c^\star(\eta)$, and by integrating with respect to x, we note that

$$\int h(x) \exp\{\eta t(x)\}\, dx = \exp\{\kappa(\eta)\}$$

for $\eta \in \mathcal{H}$ as given in lectures. Thus, for s in a suitable neighbourhood of zero, we have

$$\begin{aligned}
M_T(s) &= \mathbb{E}_{f_X}[e^{st(X)}] = \int e^{st(x)} h(x) \exp\{\eta t(x) - \kappa(\eta)\}\, dx \\
&= \exp\{-\kappa(\eta)\} \int h(x) \exp\{t(x)(\eta + s)\}\, dx = \exp\{-\kappa(\eta)\}\exp\{\kappa(\eta + s)\}
\end{aligned}$$

as $\eta \in \mathcal{H} \Longrightarrow \eta + s \in \mathcal{H}$ for s small enough, as $\mathcal{H}$ is open. Hence, as $K_T(s) = \log M_T(s)$,

$$K_T(s) = \kappa(\eta + s) - \kappa(\eta)$$

for $s \in (-h, h),$ some $h > 0$ as required.

(c) By inspection

$$\ell(x; \eta_1, \eta_2) = (\eta_1 - \eta_2) t(x) - (\kappa(\eta_1) - \kappa(\eta_2))$$

3 (a) By direct calculation the mgf of $Y_i = X_i^2$ is

$$M_{Y_i}(t) = \mathbb{E}_{f_{X_i}}[e^{tX_i^2}] = \int_{-\infty}^{\infty} e^{tx^2} \left(\frac{1}{2\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(x - \mu_i)^2\right\} dx = \left(\frac{1}{1-2t}\right)^{1/2} \exp\left\{\frac{\mu_i^2 t}{1-2t}\right\}$$

whenever $-1/2 < t < 1/2$, after completing the square in x in the exponent and integrating the result, in which the integrand is proportional to a normal pdf. Hence, using the result for independent rvs,

$$M_Y(t) = \prod_{i=1}^{r} M_{Y_i}(t) = \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\frac{\theta t}{1-2t}\right\}$$

where $\theta = \sum\limits_{i=1}^{r} \mu_i^2$.

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*The distribution of Y here is the **non-central Chisquared distribution** with r degrees of freedom and non-centrality parameter μ.*

(b) Many possible routes to compute the result. Could differentiate the mgf, or use direct calculation, or differentiate the cumulant generating function three times and evaluate at zero;

$$K_Y(t) = \log M_Y(t) = -\frac{r}{2}\log(1-2t) + \frac{\theta t}{1-2t}$$

so

$$K_Y^{(1)}(t) = \frac{r}{1-2t} + \frac{(1-2t)\theta + 2\theta t}{(1-2t)^2} = \frac{r}{1-2t} + \frac{\theta}{(1-2t)^2}$$

so that $\mu = \mathbb{E}_{f_Y}[Y] = K_Y^{(1)}(0) = r + \theta$.

$$K_Y^{(2)}(t) = \frac{2r}{(1-2t)^2} + \frac{4\theta}{(1-2t)^3}$$

so that $\sigma^2 = \text{Var}_{f_Y}[Y] = K_Y^{(2)}(0) = 2r + 4\theta = 2(r + 2\theta)$. Finally,

$$K_Y^{(3)}(t) = \frac{8r}{(1-2t)^3} + \frac{24\theta}{(1-2t)^4}$$

so that

$$\mathbb{E}_{f_Y}[(Y-\mu)^3] = K_Y^{(3)}(0) = 8r + 24\theta$$

yielding that

$$\varsigma = \frac{\mathbb{E}_{f_Y}[(Y-\mu)^3]}{\sigma^3} = \frac{8r + 24\theta}{(2r+4\theta)^{3/2}} = \frac{2^{3/2}(r+3\theta)}{(r+2\theta)^{3/2}}$$

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It is easy to verify that $K_X^{(3)}(0) = \mathbb{E}_{f_X}[(X-\mu)^3]$ by direct evaluation, complementing the results that $K_X^{(1)}(0) = \mathbb{E}_{f_X}[X]$ and $K_X^{(2)}(0) = \mathbb{E}_{f_X}[(X-\mu)^2]$.

4 (a) By iterated expectation, using the formula sheet to quote expectations for Gamma and Poisson

$$\mathbb{E}_{f_X}[X] = \mathbb{E}_{f_N}[\mathbb{E}_{f_{X|N}}[X|N=n]] = \mathbb{E}_{f_N}\left[\frac{N + r/2}{1/2}\right] = \frac{\mathbb{E}_{f_N}[N] + r/2}{1/2} = \frac{\lambda + r/2}{1/2} = 2\lambda + r$$

(b) By the same method of iterated expectation, for $-1/2 < t < 1/2$,

$$\begin{aligned}
M_X(t) = \mathbb{E}_{f_X}[e^{tX}] &= \mathbb{E}_{f_N}[\mathbb{E}_{f_{X|N}}[e^{tX}|N=n]] = \mathbb{E}_{f_N}\left[\left(\frac{1/2}{1/2-t}\right)^{N+r/2}\right] \\
&= \left(\frac{1/2}{1/2-t}\right)^{r/2}\mathbb{E}_{f_N}\left[\left(\frac{1/2}{1/2-t}\right)^{N}\right] \\
&= \left(\frac{1}{1-2t}\right)^{r/2} G_N\left(\frac{1}{1-2t}\right) \\
&= \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\lambda\left(\frac{1}{1-2t} - 1\right)\right) \\
&= \left(\frac{1}{1-2t}\right)^{r/2} \exp\left\{\frac{2\lambda t}{1-2t}\right\}
\end{aligned}$$

*The distribution of Y here is again the **non-central Chisquared distribution** with r degrees of freedom and non-centrality parameter λ, identical to the form found in Q3 (a).*

5 By iterated expectation

$$\mathbb{E}_{f_{X_1}}[X_1] = \mathbb{E}_{f_M}\left[\mathbb{E}_{f_{X_1|M}}[X_1|M=m]\right] = \mathbb{E}_{f_M}[M] = \mu$$

and

$$\mathbb{E}_{f_{X_1}}[X_1^2] = \mathbb{E}_{f_M}\left[\mathbb{E}_{f_{X_1|M}}[X_1^2|M=m]\right] = \mathbb{E}_{f_M}\left[M^2+\sigma^2\right] = \mu^2+\tau^2+\sigma^2$$

so that

$$\text{Var}_{f_{X_1}}[X_1] = \mathbb{E}_{f_{X_1}}[X_1^2] - \{\mathbb{E}_{f_{X_1}}[X_1]\}^2 = \tau^2+\sigma^2.$$

By symmetry

$$\mathbb{E}_{f_{X_2}}[X_2] = \mu \qquad \text{Var}_{f_{X_2}}[X_2] = \tau^2+\sigma^2.$$

Now,

$$\mathbb{E}_{f_{X_1,X_2}}[X_1X_2] = \mathbb{E}_{f_M}\left[\mathbb{E}_{f_{X_1,X_2|M}}[X_1X_2|M=m]\right] = \mathbb{E}_{f_M}\left[\mathbb{E}_{f_{X_1|M}}[X_1|M=m]\times\mathbb{E}_{f_{X_2|M}}[X_2|M=m]\right]$$

by conditional independence. Therefore

$$\mathbb{E}_{f_{X_1,X_2}}[X_1X_2] = \mathbb{E}_{f_M}[M\times M] = \mathbb{E}_{f_M}\left[M^2\right] = \mu^2+\tau^2$$

Hence

$$\text{Cov}_{f_{X_1,X_2}}[X_1,X_2] = \mathbb{E}_{f_{X_1,X_2}}[X_1X_2] - \mathbb{E}_{f_{X_1}}[X_1]\mathbb{E}_{f_{X_2}}[X_2] = \mu^2+\tau^2-\mu^2 = \tau^2$$

and

$$\text{Corr}_{f_{X_1,X_2}}[X_1,X_2] = \frac{\text{Cov}_{f_{X_1,X_2}}[X_1,X_2]}{\sqrt{\text{Var}_{f_{X_1}}[X_1]\text{Var}_{f_{X_2}}[X_2]}} = \frac{\tau^2}{\tau^2+\sigma^2}$$

X_1 and X_2 are not independent; their covariance is non zero.

6 As

$$S_{i+1} = \sum_{j=1}^{s_i} N_{ij} + K_i$$

with all variables independent, we have immediately using the result from lectures, and properties of pgfs, that

$$G_{i+1}(t) = G_i(G_N(t))G_K(t) = G_N(G_i(t))G_K(t)$$

where G_i is the pgf of S_i.

Note that

$$G_N(G_i(t)) = G_N(G_N(G_{i-1}(t))) = \cdots = G_N(G_N(\ldots G_N(t)\ldots))$$

*iterating i times **inside**, but taking the $i-1$ **outer** computations together yields*

$$G_{i-1}(G_N(t))$$