

MATH 556 - EXERCISES 2: SOLUTIONS

1 We have $f_R(r) = 6r(1 - r)$, for $0 < r < 1$, and hence

$$F_R(r) = r^2(3 - 2r) \quad 0 < r < 1$$

with the usual cdf behaviour outside of this range.

- Circumference: $Y = 2\pi R$, so $\mathbb{Y} = (0, 2\pi)$, and from first principles, for $y \in \mathbb{Y}$,

$$F_Y(y) = P[Y \leq y] = P[2\pi R \leq y] = P[R \leq y/2\pi] = F_R(y/2\pi) = \frac{3y^2}{4\pi^2} - \frac{2y^3}{8\pi^3}$$

$$\implies f_Y(y) = \frac{6y}{8\pi^3}(2\pi - y) \quad 0 < y < 2\pi$$

- Area: $Z = \pi R^2$, so $\mathbb{Z} = (0, \pi)$, and from first principles, for $z \in \mathbb{Z}$, recalling that f_R is only positive when $0 < z < \pi$,

$$F_Z(z) = P[Z \leq z] = P[\pi R^2 \leq z] = P[R \leq \sqrt{z/\pi}] = F_R(z/2\pi) = \frac{3z}{\pi} - 2 \left\{ \frac{z}{\pi} \right\}^{3/2}$$

$$\implies f_Z(z) = 3\pi^{-3/2}(\sqrt{\pi} - \sqrt{z}) \quad 0 < z < \pi.$$

2 If $\mathbb{X}^{(2)} = (0, 1) \times (0, 1)$ is the (joint) range of vector random variable (X, Y) . We have

$$f_{X,Y}(x, y) = cx(1 - y) \quad 0 < x < 1, 0 < y < 1$$

so that

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) \quad \text{and} \quad \mathbb{X}^{(2)} = \mathbb{X} \times \mathbb{Y}$$

where \mathbb{X} and \mathbb{Y} are the ranges of X and Y respectively, and

$$f_X(x) = c_1x \quad \text{and} \quad f_Y(y) = c_2(1 - y) \quad (1)$$

for some constants satisfying $c_1c_2 = c$. Hence, the two conditions for independence are satisfied in (1), and X and Y are independent.

Secondly, we must have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \, dx dy = 1 \quad c^{-1} = \int_0^1 \int_0^1 x(1 - y) \, dx dy = 1$$

and as

$$\int_0^1 \int_0^1 x(1 - y) \, dx dy = \left\{ \int_0^1 x \, dx \right\} \left\{ \int_0^1 (1 - y) \, dy \right\} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

we have $c = 4$.

Finally, we have $A = \{(x, y) : 0 < x < y < 1\}$, and hence, recalling that the joint density is only non-zero when $x < y$, we first fix a y and integrate dx on the range $(0, y)$, and then integrate dy on the range $(0, 1)$, that is

$$\begin{aligned} P[X < Y] &= \iint_A f_{X,Y}(x, y) \, dx dy = \int_0^1 \left\{ \int_0^y 4x(1 - y) \, dx \right\} dy \\ &= \int_0^1 \left\{ \int_0^y x \, dx \right\} 4(1 - y) \, dy = \int_0^1 2y^2(1 - y) \, dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4 \right]_0^1 = \frac{1}{6} \end{aligned}$$

3 First sketch the support of the density; this will make it clear that the boundaries of the support are different for $0 < y \leq 1$ and $y > 1$.

(i) The marginal distributions are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_{1/x}^x \frac{1}{2x^2} y dy = \frac{1}{2x^2} (\log x - \log(1/x)) = \frac{\log x}{x^2} \quad 1 \leq x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \begin{cases} \int_{1/y}^{\infty} \frac{1}{2x^2 y} dx = \frac{1}{2} & 0 \leq y \leq 1 \\ \int_y^{\infty} \frac{1}{2x^2 y} dx = \frac{1}{2y^2} & 1 \leq y \end{cases}$$

(ii) Conditionals:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{x^2 y} & 1/y \leq x \text{ if } 0 \leq y \leq 1 \\ \frac{y}{x^2} & y \leq x \text{ if } 1 \leq y \end{cases}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1}{2y \log x} \quad 1/x \leq y \leq x \text{ if } x \geq 1$$

(iii) Marginal expectation of Y ;

$$E_{f_Y}[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 \frac{y}{2} dy + \int_1^{\infty} \frac{1}{2y} dy = \infty$$

as the second integral is divergent.

4 (i) We set

$$\begin{aligned} U &= X/Y \\ V &= -\log(XY) \end{aligned} \iff \begin{aligned} X &= U^{1/2} e^{-V/2} \\ Y &= U^{-1/2} e^{-V/2} \end{aligned}$$

note that, as X and Y lie in $(0, 1)$ we have $XY < X/Y$ and $XY < Y/X$, giving constraints $e^{-V} < U$ and $e^{-V} < 1/U$, so that $0 < e^{-V} < \min\{U, 1/U\}$. The Jacobian of the transformation is

$$|J(u,v)| = \begin{vmatrix} \frac{u^{-1/2} e^{-v/2}}{2} & -\frac{u^{1/2} e^{-v/2}}{2} \\ -\frac{u^{-3/2} e^{-v/2}}{2} & -\frac{u^{-1/2} e^{-v/2}}{2} \end{vmatrix} = u^{-1} e^{-v} / 2.$$

Hence

$$f_{U,V}(u,v) = u^{-1} e^{-v} / 2 \quad 0 < e^{-v} < \min\{u, 1/u\}, u > 0$$

The corresponding marginals are given below: let $g(y) = -\log(\min\{u, 1/u\})$, then

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v) dv = \int_{g(y)}^{\infty} \frac{e^{-v}}{2u} dv = \left[-\frac{e^{-v}}{2u} \right]_{g(y)}^{\infty} = \frac{\min\{u, 1/u\}}{2u} \quad u > 0$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v) du = \int_{e^{-v}}^{e^v} \frac{e^{-v}}{2u} du = \left[\frac{\log u}{2} e^{-v} \right]_{e^{-v}}^{e^v} = v e^{-v} \quad v > 0$$

(ii) Now let

$$\begin{aligned} V &= X + Y & X &= \frac{V + Z}{2} \\ Z &= X - Y & Y &= \frac{V - Z}{2} \end{aligned} \iff$$

and the Jacobian of the transformation is $1/2$. The transformed variables take values on the square A in the (V, Z) plane with corners at $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(1, -1)$ bounded by the lines $z = -v$, $z = 2 - v$, $z = v$ and $z = v - 2$. Then

$$f_{V,Z}(v, z) = \frac{1}{2} \quad (v, z) \in A$$

and zero otherwise (sketch the square A). Hence, integrating in horizontal strips in the (V, Z) plane,

$$f_Z(z) = \int_{-\infty}^{\infty} f_{V,Z}(v, z) dv = \begin{cases} \int_{-z}^{2+z} \frac{1}{2} dv = 1 + z & -1 < z \leq 0 \\ \int_z^{2-z} \frac{1}{2} dv = 1 - z & 0 < z < 1 \end{cases}$$

5 We have $K_X(t) = \log M_X(t)$, hence

$$K_X^{(1)}(t) = \frac{d}{ds} \{K_X(t)\}_{s=t} = \frac{d}{ds} \{\log M_X(t)\}_{s=t} = \frac{M_X^{(1)}(t)}{M_X(t)} \implies K_X^{(1)}(0) = \frac{M_X^{(1)}(0)}{M_X(0)} = E_{f_X}[X]$$

as $M_X(0) = 1$. Similarly

$$K_X^{(2)}(t) = \frac{M_X(t)M_X^{(2)}(t) - \{M_X^{(1)}(t)\}^2}{\{M_X(t)\}^2}$$

and hence

$$K_X^{(2)}(0) = \frac{M_X(0)M_X^{(2)}(0) - \{M_X^{(1)}(0)\}^2}{\{M_X(0)\}^2} = E_{f_X}[X^2] - \{E_{f_X}[X]\}^2$$

and hence $K_X^{(2)}(0) = \text{Var}_{f_X}[X]$

6 (i) Put $U = X/Y$ and $V = Y$; the inverse transformations are therefore $X = UV$ and $Y = V$. In terms of the multivariate transformation theorem, we have transformation functions defined by

$$g_1(t_1, t_2) = t_1/t_2 \quad g_1^{-1}(t_1, t_2) = t_1 t_2$$

$$g_2(t_1, t_2) = t_2 \quad g_2^{-1}(t_1, t_2) = t_2$$

and the Jacobian of the transformation is given by

$$|J(u, v)| = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = |v|$$

and hence

$$f_{U,V}(u, v) = f_{X,Y}(uv, v) |v| = \left(\frac{1}{2\pi}\right) \exp\left\{-\frac{1}{2}(u^2 v^2 + v^2)\right\} |v| \quad (u, v) \in \mathbb{R}^2$$

and zero otherwise, and so, for any real u ,

$$\begin{aligned}
 f_U(u) &= \int_{-\infty}^{\infty} f_{U,V}(u, v) \, dv = \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \right) \exp \left\{ -\frac{1}{2}(u^2 v^2 + v^2) \right\} |v| \, dv \\
 &= \left(\frac{1}{\pi} \right) \int_0^{\infty} v \exp \left\{ -\frac{v^2}{2}(1 + u^2) \right\} \, dv \\
 &= \left(\frac{1}{\pi} \right) \left[-\frac{1}{(1 + u^2)} \exp \left\{ -\frac{v^2}{2}(1 + u^2) \right\} \right]_0^{\infty} = \frac{1}{\pi(1 + u^2)}
 \end{aligned}$$

with the final step following by direct integration. Thus U has a *Cauchy* distribution.

- (ii) Now put $T = X/\sqrt{S/\nu}$ and $R = S$; the inverse transformations are therefore $X = T\sqrt{R/\nu}$ and $S = R$. In terms of the multivariate transformation theorem, we have transformation functions from $(X, S) \rightarrow (T, R)$ defined by

$$\begin{aligned}
 g_1(t_1, t_2) &= t_1/\sqrt{t_2/\nu} & g_1^{-1}(t_1, t_2) &= t_1\sqrt{t_2/\nu} \\
 g_2(t_1, t_2) &= t_2 & g_2^{-1}(t_1, t_2) &= t_2
 \end{aligned}$$

and the Jacobian of the transformation is given by

$$|J(t, r)| = \begin{vmatrix} \sqrt{\frac{r}{\nu}} & \frac{t}{2\sqrt{r\nu}} \\ 0 & 1 \end{vmatrix} = \left| \sqrt{\frac{r}{\nu}} \right| = \sqrt{\frac{r}{\nu}}$$

and hence

$$f_{T,R}(t, r) = f_{X,S} \left(t\sqrt{\frac{r}{\nu}}, r \right) \sqrt{\frac{r}{\nu}} = f_X \left(t\sqrt{\frac{r}{\nu}} \right) f_S(r) \sqrt{\frac{r}{\nu}} \quad t \in \mathbb{R}, r \in \mathbb{R}^+$$

and zero otherwise, and so, for any real t ,

$$\begin{aligned}
 f_T(t) &= \int_{-\infty}^{\infty} f_{T,R}(t, r) \, dr \\
 &= \int_0^{\infty} \left(\frac{1}{2\pi} \right)^{1/2} \exp \left\{ -\frac{rt^2}{2\nu} \right\} \frac{(1/2)^{(\nu/2)}}{\Gamma(\nu/2)} r^{\nu/2-1} e^{-r/2} \sqrt{\frac{r}{\nu}} \, dr \\
 &= \left(\frac{1}{2\pi} \right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\Gamma(\nu/2)} \frac{1}{\sqrt{\nu}} \int_0^{\infty} r^{(\nu+1)/2-1} \exp \left\{ -\frac{r}{2} \left(1 + \frac{t^2}{\nu} \right) \right\} \, dr \\
 &= \left(\frac{1}{2\pi} \right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\sqrt{\nu} \Gamma(\nu/2)} \left(1 + \frac{t^2}{\nu} \right)^{-(\nu+1)/2} \int_0^{\infty} z^{(\nu+1)/2-1} \exp \left\{ -\frac{z}{2} \right\} \, dz
 \end{aligned}$$

after setting

$$z = r \left(1 + \frac{t^2}{\nu} \right).$$

Hence

$$f_T(t) = \left(\frac{1}{2\pi} \right)^{1/2} \frac{(1/2)^{(\nu/2)}}{\sqrt{\nu} \Gamma(\nu/2)} \left(1 + \frac{t^2}{\nu} \right)^{-(\nu+1)/2} \frac{\Gamma((\nu+1)/2 + 1)}{(1/2)^{(\nu+1)/2}}$$

as the integrand is proportional to a Gamma pdf. Thus

$$f_T(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+t^2/\nu)^{(\nu+1)/2}}$$

which is the *Student*(ν) density.

(iii) We have that $X|Y = y \sim N(0, y^{-1})$ and $Y \sim \text{Gamma}(\nu/2, \nu/2)$. Now, we have

$$f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y) \quad x \in \mathbb{R}, y \in \mathbb{R}^+$$

and zero otherwise, and so, for any real x ,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy \\ &= \int_0^{\infty} \sqrt{\frac{y}{2\pi}} \exp\left\{-\frac{yx^2}{2}\right\} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\nu/2-1} e^{-\nu y/2} dy \\ &= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} y^{(\nu+1)/2-1} \exp\left\{-\frac{y}{2}(\nu + x^2)\right\} dy \\ &= \frac{1}{\sqrt{2\pi}} \frac{\left(\frac{\nu}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left(\frac{1}{2}(\nu + x^2)\right)^{(\nu+1)/2}} \end{aligned}$$

as the integrand is proportional to a Gamma pdf. Therefore f_X is given by

$$f_X(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\pi\nu}\right)^{1/2} \frac{1}{(1+x^2/\nu)^{(\nu+1)/2}}$$

which is again the *Student*(ν) density.

Exercise 6 give the two alternative ways of specifying the Student-t distribution, either as a function of independent Normal and Gamma/Chi-squared variables, or as the marginal obtained by “scale-mixing” a Normal distribution by a Gamma distribution (that is, rather than having a fixed variance $\sigma^2 = 1/Y$; we regard Y as a *random variable* having a Gamma distribution, so that (X, Y) have a joint distribution

$$f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y)$$

from which we calculate $f_X(x)$ by integration.