

MATH 556 - EXERCISES 5

These exercises are not for assessment

1. Using the Central Limit Theorem, construct Normal approximations to probability distribution of a random variable X having
 - (i) a Binomial distribution, $X \sim \text{Binomial}(n, \theta)$
 - (ii) a Poisson distribution, $X \sim \text{Poisson}(\lambda)$
 - (iii) a Negative Binomial distribution, $X \sim \text{NegBinomial}(n, \theta)$
 - (iv) a Gamma distribution, $X \sim \text{Gamma}(\alpha, \beta)$

In the following questions, use the following results concerning extreme *order statistics*; let Y_n and Z_n correspond to the *maximum* and *minimum* order statistics derived from random sample X_1, \dots, X_n from population with cdf F_X , that is

$$Y_n = \max \{X_1, \dots, X_n\} \quad Z_n = \min \{X_1, \dots, X_n\}.$$

Then the cdfs of Y_n and Z_n are given by

$$F_{Y_n}(y) = \{F_X(y)\}^n \quad F_{Z_n}(z) = 1 - \{1 - F_X(z)\}^n.$$

2. Suppose $X_1, \dots, X_n \sim \text{Uniform}(0, 1)$, that is

$$F_X(x) = x \quad 0 \leq x \leq 1$$

Find the cdfs of Y_n and Z_n , and the limiting distributions as $n \rightarrow \infty$.

3. Suppose X_1, \dots, X_n have cdf

$$F_X(x) = 1 - x^{-1} \quad x \geq 1$$

Find the cdfs of Z_n and $U_n = Z_n^n$, and the limiting distributions of Z_n and U_n as $n \rightarrow \infty$.

4. Suppose X_1, \dots, X_n have cdf

$$F_X(x) = \frac{1}{1 + e^{-x}} \quad x \in \mathbb{R}$$

Find the cdfs of Y_n and $U_n = Y_n - \log n$ and the limiting distributions of Y_n and U_n as $n \rightarrow \infty$.

5. Suppose X_1, \dots, X_n have cdf

$$F_X(x) = 1 - \frac{1}{1 + \lambda x} \quad x > 0$$

Find the cdfs of Y_n and Z_n , and the limiting distributions as $n \rightarrow \infty$. Find also the cdfs of $U_n = Y_n/n$ and $V_n = nZ_n$, and the limiting distributions of U_n and V_n as $n \rightarrow \infty$.

6. Suppose $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$ are independent random variables. Let

$$M_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Show that $M_n \xrightarrow{p} \lambda$ as $n \rightarrow \infty$. If random variable T_n is defined by $T_n = e^{-M_n}$, show that $T_n \xrightarrow{p} e^{-\lambda}$, and find an approximation to the probability distribution of T_n as $n \rightarrow \infty$.

7. For the following sequences of random variables, $\{X_n\}$, decide whether the sequence converges in *mean-square* (r th mean for $r = 2$) or *in probability* as $n \rightarrow \infty$.

(a) $X_n = \begin{cases} 1 & \text{with prob. } 1/n \\ 2 & \text{with prob. } 1 - 1/n \end{cases}$

(b) $X_n = \begin{cases} n^2 & \text{with prob. } 1/n \\ 1 & \text{with prob. } 1 - 1/n \end{cases}$

(c) $X_n = \begin{cases} n & \text{with prob. } 1/\log n \\ 0 & \text{with prob. } 1 - 1/\log n \end{cases}$

Almost sure convergence and the Borel-Cantelli Lemma.

8. Consider the sequence of random variables defined for $n = 1, 2, 3, \dots$ by

$$X_n = I_{[0, n^{-1})}(U_n)$$

where U_1, U_2, \dots are a sequence of independent $Uniform(0, 1)$ random variables, and I_A is the indicator function for set A

$$I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Does the sequence $\{X_n\}$ converge

- (a) almost surely ?
(b) in r^{th} mean for $r = 1$?

[Hint: Consider the events $A_n \equiv (X_n \neq 0)$ for $n = 1, 2, \dots$]

9. Let $Z \sim Uniform(0, 1)$, and define a sequence of random variables $\{X_n\}$ by

$$X_n = nI_{[1-n^{-1}, 1)}(Z) \quad n = 1, 2, \dots$$

where, for set A

$$I_A(Z) = \begin{cases} 1 & Z \in A \\ 0 & Z \notin A \end{cases}$$

that is, I_A is the indicator random variable associated with the set A .

Does the sequence $\{X_n\}$ converge in any mode to any limit random variable ? Justify your answer.

10. Suppose, for $n = 1, 2, \dots$, $X_n \sim Bernoulli(p_n)$ are a sequence of independent random variables where

$$P[X_n = 1] = p_n = \frac{1}{\sqrt{n}}.$$

Does $P[X_n = 1 \text{ infinitely often}] = 1$? Justify your answer.